Obstacles for Magnetic Hamiltonians: SSF near Landau levels

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with G. Raikov

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In honor of André Martinez

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Schrödinger operators with constant magnetic field $B = (0, 0, b)$:

$$H_0 := -\left(\nabla^A\right)^2 = \left(D_1 + \frac{b}{2}x_2\right)^2 + \left(D_2 - \frac{b}{2}x_1\right)^2 + D_3^2$$

$$D_j := -i\frac{\partial}{\partial x_j}, \quad \nabla_j^A := \nabla_{x_j} - iA_j, \quad A = (\tfrac{b}{2}x_2, -\tfrac{b}{2}x_1, 0)$$

H_0 is s.a. on the Magnetic Sobolev spaces $\mathfrak{D}(H_0) := \mathrm{H}_A^2(\mathbb{R}^3)$

$$\|u\|^2_{\mathrm{H}_A^s(\Omega)} := \sum_{\alpha\in\mathbb{Z}_+^3 : 0\leq|\alpha|\leq s} \int_\Omega |(\nabla^A)^\alpha u|^2 \, dx.$$

$\Omega_{\rm in} \subset \mathbb{R}^3$: a connected regular bounded domain. $\Omega_{\rm ex} := \mathbb{R}^3 \setminus \Omega_{\rm in}$, $\Gamma := \partial\Omega_{\rm in} = \partial\Omega_{\rm ex}$, ν: outward normal vector at Γ (w.r.t. $\Omega_{\rm in}$)

Dirichlet realization on Ω_j: $H_{+,j}$, $j = \mathrm{ex}, \mathrm{in}$, restriction of H_0 on

$$\mathfrak{D}(H_{+,j}) := \left\{u \in \mathrm{H}_A^2(\Omega_j) \,|\, u_{|\Gamma} = 0\right\}.$$

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$$H_\pm := H_{\pm,\mathrm{in}} \oplus H_{\pm,\mathrm{ex}}.$$

Boundary condition on Γ: " + ": Dirichlet " − ": Neumann

We have:

$$H_- \leq H_0 \leq H_+$$

$\sigma(H_{\pm,\mathrm{in}}) = \sigma_{disc}(H_{\pm,\mathrm{in}}) \subset (0, +\infty)$
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Recall: spectrum of H_0:

- The Landau Hamiltonian, on $L^2(\mathbb{R}^2)$:

$$H_{0,\perp} = \left(i\frac{\partial}{\partial x_1} - \frac{bx_2}{2}\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{bx_1}{2}\right)^2 \approx (D_{y_1}^2 + y_1^2) \otimes I_{y_2}$$

Spectrum: b 3b 5b 7b 9b

$$\sigma(H_{0,\perp}) = \sigma_{\mathrm{ess}}(H_{0,\perp}) = \{\Lambda_j = b(2j+1);\ j \in \mathbb{N}\},$$

$\Lambda_j = b(2j+1)$: **Landau Level**, mult.$(\Lambda_j) = \infty$

- The 3D Magnetic Schrödinger operator, on $L^2(\mathbb{R}^3)$:

$$H_0 = H_{0,\perp} - \frac{\partial^2}{\partial x_3^2}$$

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Goal: measure of the influence of the obstacle on the spectrum near Landau levels

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The Spectral Shift Function (SSF) for the pair $(H_{\pm}, H_0)$:
In the sense of the distributions

$$\xi(E; H_{\pm}, H_0) := -\mathrm{Tr}\left(\mathbb{1}_{(-\infty,E)}(H_{\pm}) - \mathbb{1}_{(-\infty,E)}(H_0)\right)$$

For a.e. $E \in [b, \infty) = \sigma_{\mathrm{ac}}(H_0)$, *the Birman-Krein formula* implies

$$\det S(E; H_{\pm}, H_0) = e^{-2\pi i \xi(E; H_{\pm}, H_0)}$$

where $S(E; H_{\pm}, H_0)$ is *the scattering matrix*

Proposition

The SSF coincide a.e. with $\tilde{\xi}(\cdot; H_{\pm}, H_0)$ which is :
– bounded on every compact subset of $(0, \infty) \setminus b(2\mathbb{Z}_+ + 1)$
– continuous on $(0, \infty) \setminus (\sigma_p(H_{\pm}) \cup b(2\mathbb{Z}_+ + 1))$

Question 1 Behavior of $\xi(E; H_{\pm}, H_0)$ as $E \to \Lambda_q = b(2q + 1)$?
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II) Singularities of the SSF near Λ_q:

Remark: $\xi(E;H_\pm,H_0) := -\mathrm{Tr}\left(\mathbb{1}_{(-\infty,E)}(H_\pm) - \mathbb{1}_{(-\infty,E)}(H_0)\right)$

$$= -\mathrm{Tr}\left(\mathbb{1}_{(-\infty,E)}(H_{\pm,\mathrm{in}}) \oplus \mathbb{1}_{(-\infty,E)}(H_{\pm,\mathrm{ex}}) - \mathbb{1}_{(-\infty,E)}(H_0)\right)$$

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Since the spectrum of $H_{\pm,\mathrm{in}}$ is discret, near each Λ_q

$$\mathrm{Tr}\left(\mathbb{1}_{(-\infty,E)}(H_{\pm,\mathrm{in}})\right) = O(1)$$

Then as $E \to \Lambda_q$

$$\xi(E;H_\pm,H_0) = \xi(E;H_{\pm,\mathrm{ex}},H_0) + O(1)$$

$\Longrightarrow$ No influence of the interior problem

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For $\lambda \in (0, e^{-1})$ set $\ln_2(\lambda) := \ln|\ln\lambda|$, $\Phi_0(\lambda) := \frac{|\ln\lambda|}{\ln_2(\lambda)}$

Theorem 1

Let $q \in \mathbb{Z}_+$. Then we have

$$\xi(\Lambda_q - \lambda; H_+, H_0) = O(1),$$

$$\xi(\Lambda_q - \lambda; H_-, H_0) = -\frac{1}{2}\Phi_0(\lambda)(1 + o(1)),$$

$$\xi(\Lambda_q + \lambda; H_\pm, H_0) = \pm\frac{1}{4}\Phi_0(\lambda)(1 + o(1)),$$

as $\lambda \downarrow 0$.

Remark: The main contribution, $\Phi_0(\lambda)$, is independent of Ω_{in}.

Previous results: 2D case Perturbations of the Landau Ham.,
$H_{0,\perp} = (D_1 - \frac{bx_2}{2})^2 + (D_2 + \frac{bx_1}{2})^2$, $H_\perp = H_{0,\perp} + V$ or $H_{\perp,\pm}$
If $V > 0$ or $H_{\perp,+}$: infinitely many eig. above each Landau level
If $V < 0$ or $H_{\perp,-}$: infinitely many eig. below each Landau level

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xxxxx xx|xxxx x xxxxx|x xxxx xxxx|xxxx xxxxx|x xxx xxxx|x xx xx x

On $\mathbb{R}^2 \setminus \mathcal{O}_{\rm in}$ or for $V = \pm \mathbf{1}_{\mathcal{O}_{\rm in}}$, $\mathcal{O}_{\rm in} \subset \mathbb{R}^2$ bounded. Near Λ_q:
$\mathcal{N}_\pm(\lambda) := \#\{eig.(H_\perp) \in \Lambda_q \pm (\lambda, r_0)\}$, $0 < \lambda < r_0 < 2b$,

Theorem (RaWa02, PuRo07, Pe07, GoKaPe14)

$\mathcal{N}_\pm(\lambda) \sim \Phi_0(\lambda) := |\ln \lambda| \, (\ln |\ln \lambda|)^{-1}, \qquad \text{as } \lambda \searrow 0$

For non compactly supp. perturb.: Sobolev '84, Tamura '88, Raikov '90, Ivrii '98, Raik.-Warzel '02, Melgard-Rozenblum '03

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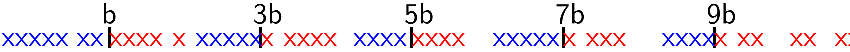

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Known results : 3D case

H: Perturbation of H_0 by

- potentials ($V \to 0$ at ∞):
  Av-He-Si '78, Fe-Ra'04, Bo-B.-Ra '07, '10
- by obstacles: B-Sa'16

The perturbation may generate :

- Accumulation of discrete eigenvalues to the first Landau Level
- Singularities of the Spectral Shift Function (Scattering Phase) at the Landau Levels: $SSF(\Lambda_q \pm \lambda) \to \infty$ as $\lambda \searrow 0$.
- Accumulation of resonances or embbeded eigenvalues to Λ_q

$$H_V = H_0 + V, \qquad V \longrightarrow 0 \text{ at } \infty$$

1) EIGENVALUES:

For $V \leq 0$, $V(X_\perp, x_3) = V(|X_\perp|, x_3)$, $X_\perp = (x_1, x_2)$ [Avron-Herbst-Simon '78]

$$-2b < V(X_\perp, x_3) \leq -C\mathbf{1}_K(X_\perp)v(x_3), \; K \subset \mathbb{R}^2, \; C > 0$$

$*$ If $v(x_3) = \mathbf{1}_{\tilde{K}}(x_3)$: at least one eig. in $(\Lambda_{q-1}, \Lambda_q)$, $q \in \mathbb{N}$
$*$ If $v(x_3) = \langle x_3 \rangle^{-\alpha}$, $\alpha \in (0,2)$: $\exists (\lambda_{q,j})_{q,j}$ eig. of H_V, $\lambda_{q,j} \to \Lambda_q$

For $0 \leq V \leq C < X_\perp >^{-m_\perp} < x_3 >^{-m_3}, m_\perp > 0, m_3 > 2$ only a discret set of embbeded eig. [Bony-B.-Raikov '07, '10]

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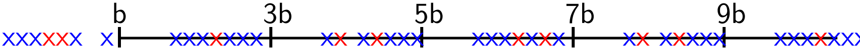

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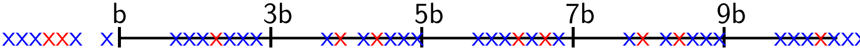

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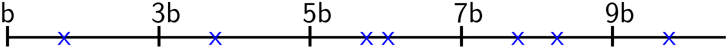

Compact. supp. pertub. of H_0 ($H_\pm = H_0 \pm V$, $V \geq 0$ or $H_\pm$ with B.C.)

2) SSF: (Fe-Ra'04, B-Ra'19)

Assume $\frac{1}{C}\mathbf{1}_{K_0} \leq V \leq C\mathbf{1}_{K_1}$, $K_0 \subset K_1 \subset \mathbb{R}^3$ bounded

Then, near a fixed Landau level Λ_q :

- For the pair (H_-, H_0), as $\lambda \downarrow 0$,

  $\xi(\Lambda_q - \lambda) \sim -\frac{|\ln \lambda|}{2 \ln |\ln \lambda|}$ $\quad \Lambda_q \quad$ $\xi(\Lambda_q + \lambda) \sim \frac{|\ln \lambda|}{4 \ln |\ln \lambda|}$

- For the pair (H_+, H_0), as $\lambda \downarrow 0$,

  $\xi(\Lambda_q - \lambda) = O(1)$ $\quad \Lambda_q \quad$ $\xi(\Lambda_q + \lambda) \sim \frac{|\ln \lambda|}{4 \ln |\ln \lambda|}$

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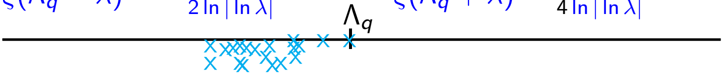
$\#\{z \ Res.;\ \mathrm{dist}(z, \Lambda_q) \geq r\} \sim \frac{|\ln r|}{\ln|\ln r|}$, $r \downarrow 0$, Compact. supp. pertub. of H_0 ($H_\pm = H_0 \pm V, V \geq 0$ or $H_\pm$ with B.C.)

Then, near a fixed Landau level Λ_q :

- For the pair (H_-, H_0), as $\lambda \downarrow 0$,

$$\xi(\Lambda_q - \lambda) \sim -\frac{|\ln \lambda|}{2 \ln |\ln \lambda|} \qquad \xi(\Lambda_q + \lambda) \sim \frac{|\ln \lambda|}{4 \ln |\ln \lambda|}$$



- For the pair (H_+, H_0), as $\lambda \downarrow 0$,

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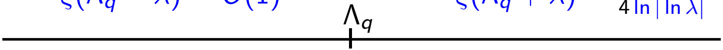

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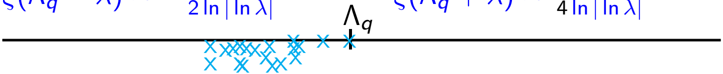

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On a 2nd sheet where resonances are defined

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Dependence on the domain in the 2D case

Comp. Supp. perturbations of the Landau Ham. $H_{0,\perp}$

Filonov-Pushnitski '06

$\{\nu_{k,q}\}_{k\in\mathbb{Z}_+}$ eig. of the s.a., compact Toeplitz operator $p_q \mathbb{1}_{\mathcal{O}} p_q$,

p_q: proj. onto $\mathrm{Ker}(H_{0,\perp} - \Lambda_q)$

$$\ln \nu_{k,q} = -k \ln k + (\mathfrak{C}(\mathcal{O}) - \ln 2)\, k + o(k), \quad k \to \infty,$$

$\mathfrak{C}(\mathcal{O}) := \ln \left(eb\, \mathrm{Cap}(\overline{\mathcal{O}})^2 \right)$

$\mathrm{Cap}(K)$: *the logarithmic capacity* of K : $\mathrm{Cap}(K) := e^{-\mathcal{I}(K)}$ where

$$\mathcal{I}(K) := \inf_{\mu \in \mathfrak{M}(K)} \int_{K\times K} \ln |x-y|^{-1} d\mu(x) d\mu(y).$$

Consequence:

Filonov-Pushnitski '06 $(H_{0,\perp} \pm \mathbb{1}_{\mathcal{O}})$,
Pu.-Roz '07/GoKaPe '14 $(H_{\perp,\pm}$: Diri./Neu. on $\partial\mathcal{O}$, $\mathcal{O} \subset \mathbb{R}^2)$

$\lambda_{k,q}$ eigenvalues of $H_{0,\perp} \pm \mathbb{1}_{\mathcal{O}}$ (or $H_{\perp,\pm}$) near Λ_q satisfy:

$$\ln\left(\pm(\lambda_{k,q} - \Lambda_q)\right) = -k \ln k + (\mathfrak{C}(\mathcal{O}) - \ln 2)\, k + o(k), \quad k \to \infty,$$

for $\mathcal{N}_\pm(\lambda) := \#\{\lambda_{k,q} \in \Lambda_q \pm (\lambda, r_0)\}$, $0 < \lambda < r_0 < 2b$, we have

$$\boxed{\mathcal{N}_\pm(\lambda) = \Phi_1(\lambda, \mathfrak{C}(\mathcal{O}) - \ln 2) + o\left(\frac{|\ln \lambda|}{\ln_2(\lambda)^2}\right)} \quad \lambda \searrow 0$$

$\Phi_1(\lambda; C) := \Phi_0(\lambda)\left(1 + \frac{\ln_3(\lambda)}{\ln_2(\lambda)} + \frac{C}{\ln_2(\lambda)}\right)$, $\ln_3(\lambda) := \ln \ln_2(\lambda)$,

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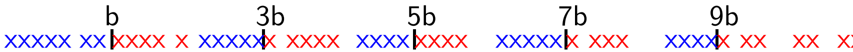

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Asymptotic behavior of the SSF near Λ_q in the 3D case

Let $\mathcal{O}_{\rm in}$ be the projection of $\Omega_{\rm in}$ onto the plane $\perp$ to $\vec{B}$:

$$\mathcal{O}_{\rm in} := \pi_\perp(\Omega_{\rm in}) := \left\{X_\perp \in \mathbb{R}^2 \mid \exists\, x_\| \in \mathbb{R} \text{ s.t. } (X_\perp, x_\|) \in \Omega_{\rm in}\right\}$$

Theorem 2

Let $q \in \mathbb{Z}_+$. Under assumption $\mathcal{A}$ (see after) for $\Omega_{\rm in}$, we have

(i) $\xi(\Lambda_q - \lambda; H_-, H_0) = -\frac{1}{2}\,\Phi_1(\lambda; \mathfrak{C}(\mathcal{O}_{\rm in})) + o\left(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\right)$

(ii) $\xi(\Lambda_q + \lambda; H_\pm, H_0) = \pm\frac{1}{4}\,\Phi_1(\lambda; \mathfrak{C}(\mathcal{O}_{\rm in})) + o\left(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\right)$

as $\lambda \downarrow 0$.

$\mathfrak{C}(\mathcal{O}) := \ln\left(eb\,\mathrm{Cap}(\overline{\mathcal{O}})^2\right)$

$\Phi_1(\lambda; C) := \Phi_0(\lambda)\left(1 + \frac{\ln_3(\lambda)}{\ln_2(\lambda)} + \frac{C}{\ln_2(\lambda)}\right)$, $\Phi_0(\lambda) := \frac{|\ln r|}{\ln_2(\lambda)}$

$\ln_3(\lambda) := \ln\ln_2(\lambda)$, $\ln_2(\lambda) := \ln\ln(\lambda)$,

Definition: $\Omega_{\rm in}$ **satisfies assumption $\mathcal{A}$ if:**
$\exists$ **adjacent sequences** $\{\Omega_{j,<}\}_{j\in\mathbb{N}} \nearrow$ and $\{\Omega_{j,>}\}_{j\in\mathbb{N}} \searrow$ s.t.:
(i) $\overline{\Omega}_{j,<} \subset \Omega_{\rm in} \subset \overline{\Omega}_{\rm in} \subset \Omega_{j,>}$, $j \in \mathbb{N}$;
(ii) For $\mathcal{O}_{j,<} := \pi_\perp(\Omega_{j,<})$ and $\mathcal{O}_{j,>} := \pi_\perp(\Omega_{j,>})$, $j \in \mathbb{N}$, $\partial\mathcal{O}_{j,<}$ and $\partial\mathcal{O}_{j,>}$ are Lipschitz
(iii) For $\mathcal{O}_{\rm in} := \pi_\perp(\Omega_{\rm in})$;

$$\lim_{j\to\infty} \mathrm{Cap}(\overline{\mathcal{O}}_{j,<}) = \lim_{j\to\infty} \mathrm{Cap}(\overline{\mathcal{O}}_{j,>}) = \mathrm{Cap}(\overline{\mathcal{O}}_{\rm in}),$$

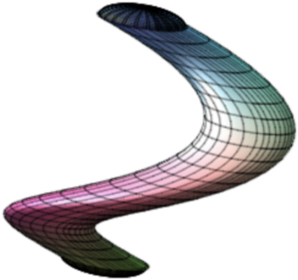
(iv) For any $j \in \mathbb{N}$,

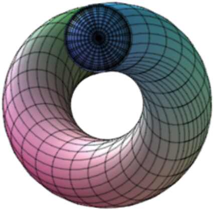
$$\inf_{X_\perp \in \mathcal{O}_{j,<}} \int_{\mathbb{R}} \mathbb{1}_{\Omega_{j+1,<}}(X_\perp, x_\parallel)\, dx_\parallel > 0, \quad j \in \mathbb{N}.$$

Evidently, any ball (or ellipsoid) in $\mathbb{R}^3$ satisfies assumption $\mathcal{A}$.

Exemples

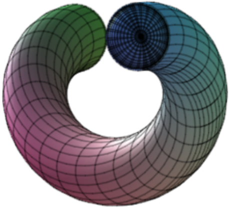
$\Omega_{\rm in}$





$\mathcal{O}_{\rm in}$ smooth

$\mathcal{O}_{\rm in}$ not Lipschitz



III) Ideas of the proof

We have

$$\xi(E; H_\pm, H_0) = -\xi(\frac{1}{E}; H_\pm^{-1}, H_0^{-1})$$

$H_\pm^{-1}$ and H_0^{-1} defined in $L^2(\mathbb{R}^3)$ and $H_\pm^{-1} = H_0^{-1} \mp V_\pm$ with

$$V_+ := H_0^{-1} - H_+^{-1} \geq 0, \qquad V_- := H_-^{-1} - H_0^{-1} \geq 0$$

Step 1: Pushnitski's representation formula (Push. '97)
$\Longrightarrow$ study of the Birman-Schwinger operator:

$$V_\pm^{\frac{1}{2}}\left(H_0^{-1} - E^{-1}\right)^{-1} V_\pm^{\frac{1}{2}} = EV_\pm^{\frac{1}{2}} H_0(E - H_0)^{-1} V_\pm^{\frac{1}{2}}$$
$$= -EV_\pm - E^2 V_\pm^{\frac{1}{2}}(H_0 - E)^{-1} V_\pm^{\frac{1}{2}}$$

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Following Fe-Ra'04, as $\lambda \downarrow 0$, we have

Proposition 1

$$\xi(\Lambda_q - \lambda; H_+, H_0) = O(1),$$

$$-\mathrm{Tr}\, \mathbb{1}_{[1,+\infty)} \left(\frac{\mathcal{M}_q^-}{(1-\epsilon)2\sqrt{\lambda}} \right) + O(1) \le \xi(\Lambda_q - \lambda; H_-, H_0) \le -\mathrm{Tr}\, \mathbb{1}_{[1,+\infty)} \left(\frac{\mathcal{M}_q^-}{(1+\epsilon)2\sqrt{\lambda}} \right) + O(1)$$

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$(0 \neq$ e.v.$(TT^*) =$ e.v. $(T^*T))$

Step 3: Localization near Γ: $\forall\ w \in C_c^{\infty}(\mathbb{R}^3)$, $w = 1$ near Γ,

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Proposition 2

Let $\Omega_< \subset \Omega_{\rm in} \subset \Omega_>$ as in Assumption $\mathcal{A}$
Let $\mathcal{E}_q(\Omega_>) = \{f \in L^2(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3);\ (H_0 - \Lambda_q)f = 0 \text{ on } \Omega_>\}$
Then there exists $\mathcal{L}_q$ a finite codimension subspaces of $\mathcal{E}_q(\Omega_>)$ and $C > 1$ s. t. $\forall f \in \mathcal{L}_q$,

$$\frac{1}{C}\langle f, \mathbf{1}_{\Omega_<} f\rangle_{L^2(\mathbb{R}^3)} \leq \langle H_0 wf, V_\pm H_0 wf\rangle_{L^2(\mathbb{R}^3)} \leq C\langle f, \mathbf{1}_{\Omega_>} f\rangle_{L^2(\mathbb{R}^3)}$$

$\Longrightarrow$ Theorem follows using that

$$c\, \mathbb{1}_{\mathcal{O}_{j,<}}(X_\perp) \leq \int_{\mathbb{R}} \mathbb{1}_{\Omega_{j+1,<}}(X_\perp, x_\parallel)\, dx_\parallel$$

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$\Longrightarrow$Theorem follows using that

$$c\,\mathbb{1}_{\mathcal{O}_{j,<}}(X_\perp) \leq \int_{\mathbb{R}} \mathbb{1}_{\Omega_{j+1,<}}(X_\perp, x_\parallel)\, dx_\parallel$$

$$\int_{\mathbb{R}} \mathbb{1}_{\Omega_{j+1,>}}(X_\perp, x_\parallel)\, dx_\parallel \leq C\mathbb{1}_{\mathcal{O}_{j+1,>}}(X_\perp)$$

Proof of the lower bound of Proposition 2

$$\langle H_0 wf , V_\pm H_0 wf \rangle_{L^2(\mathbb{R}^3)} \geq \frac{1}{C} \langle f , \mathbf{1}_{\Omega_<} f \rangle_{L^2(\mathbb{R}^3)}$$

For V_+:

$V_+ := H_0^{-1} - H_{+,\mathrm{in}}^{-1} \oplus H_{+,\mathrm{ex}}^{-1} = V_{+,0} - H_{+,\mathrm{in}}^{-1} \oplus 0$, with

$V_{+,0} := H_0^{-1} - 0 \oplus H_{+,\mathrm{ex}}^{-1} \geq \left(H_0^{-1} - (H_0 + \mathbb{1}_{\Omega_<})^{-1} \right)$

because

$V_{+,0} = \left(H_0^{-1} - (H_0 + \mathbb{1}_{\Omega_<})^{-1} \right) + \left((H_0 + \mathbb{1}_{\Omega_<})^{-1} - 0 \oplus H_{+,\mathrm{ex}}^{-1} \right)$

We conclude using

$H_0^{-1} - (H_0 + \mathbb{1}_{\Omega_<})^{-1} =$

$H_0^{-1} \mathbb{1}_{\Omega_<} \left(I - \mathbb{1}_{\Omega_<} (H_0 + \mathbb{1}_{\Omega_<})^{-1} \mathbb{1}_{\Omega_<} \right) \mathbb{1}_{\Omega_<} H_0^{-1} \geq H_0^{-1} \mathbb{1}_{\Omega_<} H_0^{-1}$

on a finite codim. subsp.

Proof of the lower bound of Proposition 2

$$\langle H_0 wf, V_\pm H_0 wf\rangle_{L^2(\mathbb{R}^3)} \geq \frac{1}{C}\langle f, \mathbf{1}_{\Omega_<} f\rangle_{L^2(\mathbb{R}^3)}$$

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We conclude using

$H_0^{-1} - (H_0 + \mathbb{1}_{\Omega_<})^{-1} =$

$H_0^{-1}\mathbb{1}_{\Omega_<}\left(I - \mathbb{1}_{\Omega_<}(H_0 + \mathbb{1}_{\Omega_<})^{-1}\mathbb{1}_{\Omega_<}\right)\mathbb{1}_{\Omega_<} H_0^{-1} \geq H_0^{-1}\mathbb{1}_{\Omega_<} H_0^{-1}$

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Proof of the lower bound of Proposition 2

$$\langle H_0 wf, V_\pm H_0 wf \rangle_{L^2(\mathbb{R}^3)} \geq \frac{1}{C} \langle f, \mathbf{1}_{\Omega_<} f \rangle_{L^2(\mathbb{R}^3)}$$

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on a finite codim. subsp.

For V_-:

$V_- := H_{-,\mathrm{in}}^{-1} \oplus H_{-,\mathrm{ex}}^{-1} - H_0^{-1} = V_{-,0} - \delta H_{-,\mathrm{in}}^{-1} \oplus 0$, $\delta > 0$ with $V_{-,0} := (1+\delta) H_{-,\mathrm{in}}^{-1} \oplus H_{-,\mathrm{ex}}^{-1} - H_0^{-1}$

There exists $\kappa > 0$ s.t.:

$$V_{-,0} \geq \left((H_0 - \kappa \mathbb{1}_{\Omega_<})^{-1} - H_0^{-1} \right),$$

because $V_{-,0} = \left((1+\delta) H_{-,\mathrm{in}}^{-1} \oplus H_{-,\mathrm{ex}}^{-1} - (H_0 - \kappa \mathbb{1}_{\Omega_<})^{-1} \right) + \left((H_0 - \kappa \mathbb{1}_{\Omega_<})^{-1} - H_0^{-1} \right)$,

We conclude using

$$\left((H_0 - \kappa \mathbb{1}_{\Omega_<})^{-1} - H_0^{-1} \right) = \kappa H_0^{-1} \mathbb{1}_{\Omega_<} \left(I + \kappa \mathbb{1}_{\Omega_<} (H_0 - \kappa \mathbb{1}_{\Omega_<})^{-1} \mathbb{1}_{\Omega_<} \right) \mathbb{1}_{\Omega_<} H_0^{-1} \geq \kappa H_0^{-1} \mathbb{1}_{\Omega_<} H_0^{-1}$$

For V_-:
$V_- := H_{-,\mathrm{in}}^{-1} \oplus H_{-,\mathrm{ex}}^{-1} - H_0^{-1} = V_{-,0} - \delta H_{-,\mathrm{in}}^{-1} \oplus 0$, $\delta > 0$ with
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$$\geq \kappa H_0^{-1} \mathbb{1}_{\Omega_<} H_0^{-1}$$

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We conclude using
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$\kappa H_0^{-1} \mathbb{1}_{\Omega_<} \Big(I + \kappa \mathbb{1}_{\Omega_<} (H_0 - \kappa \mathbb{1}_{\Omega_<})^{-1} \mathbb{1}_{\Omega_<} \Big) \mathbb{1}_{\Omega_<} H_0^{-1}$

$$\geq \kappa H_0^{-1} \mathbb{1}_{\Omega_<} H_0^{-1}$$

Conclusion on the 3D magnetic problem with obstacle $\Omega_{\rm in}$:

As $\lambda \downarrow 0$, the SSF satisfies in particular :

$$\xi(\Lambda_q + \lambda; H_{\pm,\rm ex}, H_0) =$$

$$\pm\frac{1}{4}\left(\frac{|\ln\lambda|}{\ln_2(\lambda)} + \frac{|\ln\lambda|\ln_3(\lambda)}{\ln_2(\lambda)^2} + \frac{|\ln\lambda|\mathfrak{C}(\mathcal{O}_{\rm in})}{\ln_2(\lambda)^2}\right) + o\left(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\right)$$

with:

$\mathfrak{C}(\mathcal{O}) := \ln\left(eb\,\mathrm{Cap}(\overline{\mathcal{O}})^2\right)$

$\mathcal{O}_{\rm in} := \pi_\perp(\Omega_{\rm in}) := \left\{X_\perp \in \mathbb{R}^2 \mid \exists\, x_\| \in \mathbb{R} \text{ s.t. } (X_\perp, x_\|) \in \Omega_{\rm in}\right\}$

Under assumption $\mathcal{A}$

Question: Is it a technical assumption? Or more fundamental?

Conclusion on the 3D magnetic problem with obstacle Ω_{in}:

As $\lambda \downarrow 0$, the SSF satisfies in particular :

$$\xi(\Lambda_q + \lambda; H_{\pm,\text{ex}}, H_0) =$$

$$\pm\frac{1}{4}\left(\frac{|\ln\lambda|}{\ln_2(\lambda)} + \frac{|\ln\lambda|\ln_3(\lambda)}{\ln_2(\lambda)^2} + \frac{|\ln\lambda|\mathfrak{C}(\mathcal{O}_{\text{in}})}{\ln_2(\lambda)^2}\right) + o\left(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\right)$$

with:

$\mathfrak{C}(\mathcal{O}) := \ln\left(eb\,\mathrm{Cap}(\overline{\mathcal{O}})^2\right)$

$\mathcal{O}_{\text{in}} := \pi_\perp(\Omega_{\text{in}}) := \left\{X_\perp \in \mathbb{R}^2 \mid \exists\, x_\parallel \in \mathbb{R} \text{ s.t. } (X_\perp, x_\parallel) \in \Omega_{\text{in}}\right\}$

Under assumption $\mathcal{A}$

Question: Is it a technical assumption? Or more fundamental?