On the multiplicity of the second eigenvalue of the Laplacian in non simply connected domains

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en l'honneur de André Martinez

Abstract

We revisit an interesting example proposed by Maria and Thomas Hoffmann-Ostenhof, together with Nikolai Nadirashvili of a bounded domain in $\mathbb{R}^2$ for which the second eigenvalue of the Dirichlet Laplacian has multiplicity 3. We also analyze carefully the first eigenvalues of the Laplacian in the case of the disk with two symmetric cracks placed on a smaller concentric disk in function of their size.
This is a common work with Thomas Hoffmann-Ostenhof, François Jauberteau and Corentin Léna.

Introduction

The motivating problem is to analyze the multiplicity of the k-th eigenvalue of the Dirichlet problem in a domain Ω in $\mathbb{R}^2$. It is for example an old result of Cheng [Ch1976], that the multiplicity of the second eigenvalue is at most 3.
In [HOHON2–1997], M and T. Hoffmann-Ostenhof and N. Nadirashvili give an example with multiplicity 3 is given as a side product of the production of a counter example to the nodal line conjecture (see also [HOHON1–1997], and the papers by S. Fournais [Fo2001] and J.B. Kennedy [K2013] who extend to higher dimensions these counter examples, introducing new methods).
This example is based on the spectral analysis of the Laplacian in domains consisting of a disc in which is introduced an interior concentric circle suitable cracks.

The initial proof contained a gap and our aim is to fill the gap and discuss extensions.
Although not needed for the positive results, we also propose a fine theoretical analysis of the spectral problem when the cracks are closing.

Main statement

The starting point for the construction of counterexamples to the nodal line conjecture is the introduction of two concentric open discs B_{R_1} and B_{R_2} with $0 < R_1 < R_2$ and the corresponding annulus $M_{R_1,R_2} = B_{R_2} \setminus \bar{B}_{R_1}$. The authors choose R_1 and R_2 such that

$$(A) \quad \lambda_1(B_{R_1}) < \lambda_1(M_{R_1,R_2}) < \lambda_2(B_{R_1}),$$

where, for $\omega \subset \mathbb{R}^2$ bounded, $\lambda_j(\omega)$ denotes the j-th eigenvalue of the Dirichlet Laplacian H in ω.

Then we introduce

$$D_{R_1,R_2} = B_{R_1} \cup M_{R_1,R_2}$$

and observe that

$$\begin{aligned}
\lambda_1(D_{R_1,R_2}) &= \lambda_1(B_{R_1}) \\
\lambda_2(D_{R_1,R_2}) &= \lambda_1(M_{R_1,R_2}) \\
\lambda_3(D_{R_1,R_2}) &= \min(\lambda_2(B_{R_1}), \lambda_2(M_{R_1,R_2}))\,.
\end{aligned}$$

If Condition (A) was important in the construction of the counter-example to the nodal line conjecture, the weaker assumption

$$(B) \quad \max(\lambda_1(B_{R_1}), \lambda_1(M_{R_1,R_2})) < \min(\lambda_2(B_{R_1}), \lambda_2(M_{R_1,R_2}))\,. \tag{1}$$

suffices for the multiplicity question. Under this condition, we have:

$$\begin{aligned} \lambda_1(D_{R_1,R_2}) &= \min(\lambda_1(B_{R_1}), \lambda_1(M_{R_1,R_2})) \\ \lambda_2(D_{R_1,R_2}) &= \max(\lambda_1(B_{R_1}), \lambda_1(M_{R_1,R_2})) \\ \lambda_3(D_{R_1,R_2}) &= \min(\lambda_2(B_{R_1}), \lambda_2(M_{R_1,R_2}))\,, \end{aligned} \tag{2}$$

and it is not excluded to consider the case
$\lambda_1(D_{R_1,R_2}) = \lambda_2(D_{R_1,R_2})$.
We actually make this choice in the numerics.

We now carve holes in ∂B_{R_1} such that D_{R_1,R_2} becomes a domain. For $N \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and $\epsilon \in [0, \frac{\pi}{N}]$, we introduce

$$\mathfrak{D}(N,\epsilon) = D_{R_1,R_2} \cup_{j=0}^{N-1} \{x \in \mathbb{R}^2 \, , \, r = R_1 \, , \, \theta \in (\frac{2\pi j}{N} - \epsilon, \frac{2\pi j}{N} + \epsilon)\} \, . \tag{3}$$

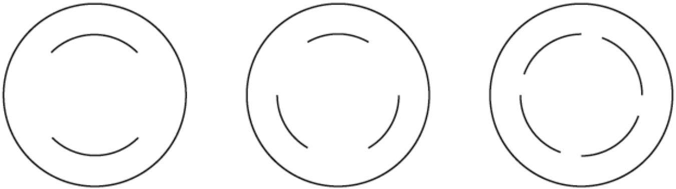


Figure: The domains with cracks for $N = 2$, $N = 3$ and $N = 4$.

The theorem stated in [HOHON2–1997] is the following:

Main Theorem

Let $N \geq 3$, then there exists $\epsilon \in (0, \frac{\pi}{N})$ such that $\lambda_2(\mathfrak{D}(N, \epsilon))$ has multiplicity 3.

The proof given in [HOHON2] can only work for even integers $N \geq 4$ (with a need for additional arguments). So we improve the result by giving an example $\Omega := \mathfrak{D}(3, \epsilon)$ where the number of components of $\partial\Omega$ equals 4, hence $N = 3$.

Remark

Our main theorem leads to the following open question:

Is there a bounded domain $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ has strictly less than 4 components so that $\lambda_2(\Omega)$ has multiplicity 3?

The natural conjecture would be that for simply connected domains Ω, $\lambda_2(\Omega)$ has at most multiplicity 2.

At the moment it is only proven for convex planar domains or for simply connected domains for which the nodal line conjecture holds true (see Lin–1987).

Symmetry spaces

We recall some basic representation theory. We consider a Hamiltonian which is the Dirichlet realization of the Laplacian in an open set Ω which is invariant by the action of the group G_N generated by the rotation g by $\frac{2\pi}{N}$.

The Hilbert space is $\mathcal{H} := L^2(\Omega, \mathbb{R})$ but it is also convenient to work in $\mathcal{H}_\mathbb{C} := L^2(\Omega, \mathbb{C})$. In this case, it is natural to analyze the eigenspaces attached to the irreducible representations of G_N.

The theory will in particular apply for the family of open sets $\Omega = \mathfrak{D}(N, \epsilon)$.

The theory is simpler for complex Hilbert spaces i.e. $\mathcal{H}_{\mathbb{C}} := L^2(\Omega, \mathbb{C})$, but the multiplicity property appears when considering operators on real Hilbert spaces, i.e $\mathcal{H} := L^2(\Omega, \mathbb{R})$.
If we work in $\mathcal{H}_{\mathbb{C}}$, we introduce for $\ell = 0, \cdots, N-1$,

$$\mathcal{B}_\ell = \{w \in \mathcal{H}_{\mathbb{C}} \mid gw = e^{2\pi i \ell / N} w\}\,. \tag{4}$$

For $\ell = 0$, this corresponds to the invariant situation.
We also observe that the complex conjugation sends $\mathcal{B}_\ell$ onto $\mathcal{B}_{N-\ell}$. Hence, except in the cases $\ell = 0$ and $\ell = \frac{N}{2}$ the corresponding eigenspace are of even dimension.
The second case appears only if N is even.

For $2\ell \neq N$, one can alternately come back to real vector spaces by introducing for $0 < \ell < \frac{N}{2}$ ($\ell \in \mathbb{N}$)

$$\mathcal{C}_\ell = \mathcal{B}_\ell \oplus \mathcal{B}_{N-\ell} \tag{5}$$

and observing that $\mathcal{C}_\ell$ can be recognized as the complexification of the real vector space $\mathcal{A}_\ell$

$$\mathcal{A}_\ell = \{u \in \mathcal{H} \mid u - 2\cos(2\ell\pi/N)gu + g^2u = 0\} \tag{6}$$

such that

$$\mathcal{C}_\ell = \mathcal{A}_\ell \otimes \mathbb{C} \tag{7}$$

where (6) follows from an easy computation based on (4).
For $\ell = 0$ and $\ell = \frac{N}{2}$ (if N is even), we define $\mathcal{A}_\ell$ by

$$\mathcal{B}_\ell = \mathcal{A}_\ell \otimes \mathbb{C}\,. \tag{8}$$

Under the invariance condition on the domain, the Dirichlet Laplacian commutes with the natural action of g in L^2. Hence we get for $0 \leq \ell \leq N/2$ a family of selfadjoint operators $H^{(\ell)}$ obtained by restriction of H to $\mathcal{A}_\ell$ (with domain $D(H) \cap \mathcal{A}_\ell$). Except for $\ell = 0$ and $\ell = \frac{N}{2}$ all the eigenspaces of $H^{(\ell)}$ have even multiplicity.

The other point is that Stollmann's theory [Sto] works for the spectrum of $H^{(\ell)}(\epsilon, N)$ associated with the Dirichlet realization $H(\epsilon, N)$ of the Laplacian in $\mathfrak{D}_{N,\epsilon}$. Hence we have continuity and monotonicity with respect to ϵ of the eigenvalues. Note also that

$$\sigma(H(\epsilon, N)) = \cup_{0 \leq \ell \leq \frac{N}{2}} \sigma(H^{(\ell)}(\epsilon, N)) .$$

When N is even, a particular role is played by $g^{\frac{N}{2}}$ which corresponds to the inversion considered in [HOHON2]. One can indeed decompose the Hilbert space $\mathcal{H}$ (or $\mathcal{H}_{\mathbb{C}}$) using the symmetry with respect to $g^{\frac{N}{2}}$ and get the decomposition

$$\mathcal{H} = \mathcal{H}^{even} \oplus \mathcal{H}^{odd} , \tag{9}$$

and

$$H(\epsilon, N) = H^{even}(\epsilon, N) \oplus H^{odd}(\epsilon, N) . \tag{10}$$

We observe that $\mathcal{A}_\ell$ belongs to $\mathcal{H}^{even}$ if ℓ is even and to $\mathcal{H}^{odd}$ if ℓ is odd.

Upper bound: the regularity assumptions in Cheng's statement revisited

In [Ch], S.Y. Cheng proved that the multiplicity of the second eigenvalue is at most 3. Cheng's proof is actually using a regularity assumption which is not satisfied by $D(N, \epsilon)$. This domain has indeed cracks and we need a description of the nodal line structure near corners or cracks. The extension can be obtained by using a paper of Helffer, Hoffmann-Ostenhof, and Terracini [HHOT–2009].

With this complementary analysis near the cracks, we can follow the main steps of the proof given in Hoffmann-Ostenhof-Michor-Nadirashvili [HOMN–1999]. This proof includes an extended version of Euler's Polyhedral formula.

Proposition Euler

Let Ω be a $C^{1,+}$-domain[1] with possibly corners of opening[2] $\alpha\pi$ for $0 < \alpha \leq 2$. If u is an eigenfunction of the Dirichlet Laplacian in Ω, $\mathcal{N}$ denotes the nodal set of u and $\mu(\mathcal{N})$ denotes the cardinality of the components of $\Omega \setminus \mathcal{N}$, i.e. the number of nodal domains, then

$$\mu(\mathcal{N}) \geq \sum_{x \in \mathcal{N} \cap \Omega} (\nu(x) - 1) + 2 \,, \tag{11}$$

where $\nu(x)$ is the multiplicity of the critical point $x \in \mathcal{N}$ (i.e. the number of lines crossing at x).

[1] $C^{1,+}$ means $C^{1,\epsilon}$ for some $\epsilon > 0$.

[2] $\alpha = 2$ corresponds to the crack case.

For a second eigenfunction $\mu(\mathcal{N}) = 2$, and the upper bound of the multiplicity by 3 comes by contradiction. Assuming that the multiplicity of the second eigenvalue is ≥ 4, one can, for any $x \in \Omega$, construct some u in the second eigenspace such that $\nu(x) \geq 2$. This gives the contradiction with Euler's formula.

Hence we have

Proposition

Let Ω be a $C^{1,+}$-domain with possibly corners of opening $\alpha\pi$ for $0 < \alpha \leq 2$. Then the multiplicity of the second eigenvalue of the Dirichlet Laplacian in Ω is not larger than 3.

Remark

An upper bound of the multiplicity by 2 is obtained by C.S. Lin when Ω is convex ([Li-1987]). Lin's theorem can be extended to the case of a simply connected domain for which the nodal line conjecture holds. If the multiplicity of the second eigenvalue is larger than 2, one can indeed find in the associated spectral space an eigenfunction whose nodal set contains a point in the boundary where two half lines hit the boundary. This will contradict either the nodal line conjecture or Courant's theorem. See also [HOHON2] for some sufficient conditions on domains for the nodal line conjecture to hold.

Proof of Main Theorem

We first observe that for the disk of radius R we have

$$\lambda_1(B_R) < \lambda_2(B_R) = \lambda_3(B_R) < \lambda_4(B_R) = \lambda_5(B_R) < \lambda_6(B_R)\,.$$

The eigenfunctions u_1 and u_6 are radial. We use this property with $R = R_2$.

Proposition

For $N \geq 3$, there exists $\epsilon \in (0, \frac{\pi}{N})$ s. t. $\lambda_2(H(\epsilon, N))$ belongs to $\sigma(H^{(\ell)}(\epsilon, N))$ for some $0 < \ell < \frac{N}{2}$ AND to $\sigma(H^{(\ell)}(\epsilon, N))$ for $\ell = 0$ or (in the case N even) $\frac{N}{2}$. In particular, the multiplicity of λ_2 for this value of ϵ is exactly 3.

Proof

Note that the condition $N \geq 3$ implies the existence of at least one $\ell \in (0, \frac{N}{2})$.

We now proceed by contradiction. Suppose the contrary. By continuity of the second eigenvalue, we should have

- either $\lambda_2(H(\epsilon, N))$ belongs to $\cup_{0<\ell<\frac{N}{2}} \sigma(H^{(\ell)}(\epsilon, N)))$ and not to
  $\sigma(H^{(0)}(\epsilon, N)) \cup \sigma(H^{(N/2)}(\epsilon, N))$ for any ϵ,
- or $\lambda_2(H(\epsilon, N))$ belongs to $\sigma(H^{(0)}(\epsilon, N)) \cup \sigma(H^{(N/2)}(\epsilon, N))$ and not to
  $\cup_{0<\ell<\frac{N}{2}} \sigma(H^{(\ell)}(\epsilon, N)))$ for any ϵ.

But, the analysis for $\epsilon > 0$ small enough shows that we should be in the first case and the analysis for ϵ close to $\frac{\pi}{N}$ that we should be in the second case. Hence a contradiction.

The analysis for $\epsilon > 0$ very small is by perturbation a consequence of the analysis of $\epsilon = 0$. Here we see that under Condition (A) $\lambda_2(D_{R_1,R_2})$ is simple and belongs to $\sigma(H^{(0)}(0, N))$.

Remark

If we only have Condition (B), we observe that the two first eigenvalues belong to $\sigma(H^{(0)}(0,N))$ and the argument is unchanged.

The analysis for ϵ close to $\frac{\pi}{N}$ is by perturbation a consequence of the analysis of $\epsilon = \frac{\pi}{N}$.

We have seen that $\lambda_2(B_{R_2})$ has multiplicity two corresponding to $\ell = 1$, hence in $\sigma(H^{(1)}(\frac{\pi}{N}, N))$.

So we have proven that for this value of ϵ the multiplicity is at least three, hence equals three by the extension of Cheng's statement.

For the specific choice of the pair (R_1, R_2), corresponding to the choice $\lambda_1(B_{R_1}) = \lambda_1(M_{R_1,R_2})$, the numerics illustrates the statement of the main theorem when $N = 3$, $N = 4$ and $N = 6$. In this case $\lambda_1(B_{R_1})$ is an eigenvalue for any ϵ.

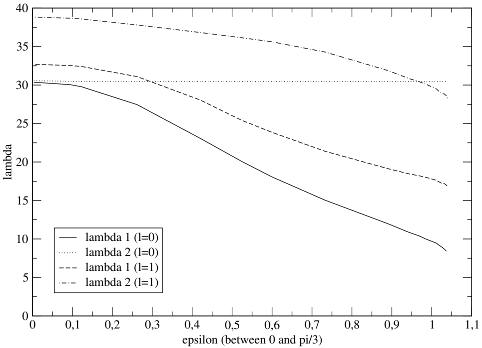


Figure: $N = 3$. Six lowest eigenvalues of the Laplacian in $\mathfrak{D}(N, \epsilon))$ in function of $\epsilon \in (0, \frac{\pi}{3})$, with $R_1 = 0.4356$, $R_2 = 1$.

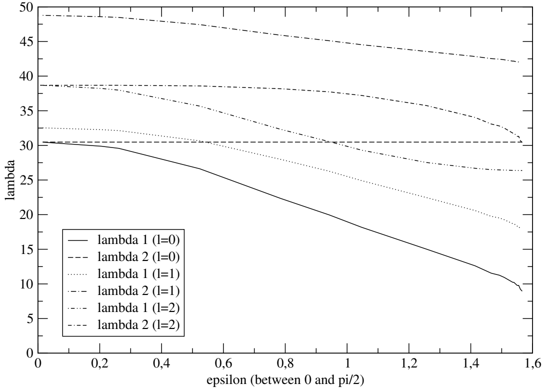


Figure: $N = 4$. Eight lowest eigenvalues of the Laplacian in $\mathfrak{D}(N, \epsilon))$ in function of $\epsilon \in (0, \frac{\pi}{2})$, with $R_1 = 0.4356$, $R_2 = 1$.

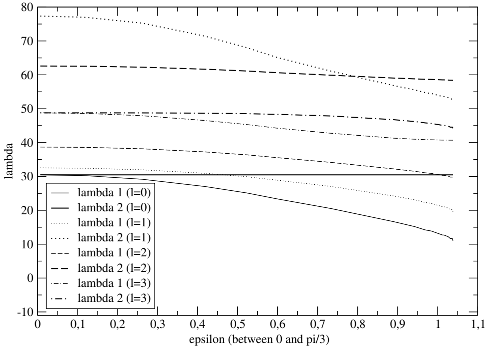


Figure: The case $N = 6$. Twelve first eigenvalues ($R_1 = 0.4356$)

Theoretical asymptotics in domains with cracks

We analyze theoretically the behavior of the eigenvalue as ϵ tends to $\frac{\pi}{2}$. This improves the general results based on Stollmann [Sto-1995].

We now fix $N = 2$ and consider $0 < R_1 < R_2$. We analyze the different spectral problems according to the symmetries. This leads us to consider on the quarter of a disk ($0 < \theta < \frac{\pi}{2}$) four different models. On the exterior circle and on the cracks, we assume the Dirichlet condition and then, according to the boundary conditions retained for $\theta = 0$ and $\theta = \pi/2$, we consider four test cases.

The four cases

- Case NND (homogeneous Neumann boundary conditions for $\theta = 0$ and $\theta = \pi/2$).
- Case DDD (homogeneous Dirichlet boundary conditions for $\theta = 0$ and $\theta = \pi/2$).
- Case NDD (homogeneous Neumann boundary conditions for $\theta = 0$ and homogeneous Dirichlet boundary conditions for $\theta = \pi/2$).
- Case DND (homogeneous Dirichlet boundary conditions for $\theta = 0$ and homogeneous Neumann boundary conditions for $\theta = \pi/2$).

This is immediately related to the problem on the cracked disk by using the symmetries with respect to the two axes. The symmetry properties lead either to Dirichlet or Neumann. We just refer to two cases.

The cases NND and DND

We use the notation

$$
\begin{array}{ll}
B^{+}_{R_2} & := B_{R_2} \cap \{x_2 > 0\}\,; \\
x_{\pm} & := (0, \pm R_1)\,; \\
\delta & := \frac{\pi}{2} - \epsilon\,; \\
K_\delta & := \{x\,;\; r = R_1, \theta \in [-\pi/2 - \delta, -\pi/2 + \delta] \cap [\pi/2 - \delta, \pi/2 + \delta]\}\,; \\
K^{+}_{\delta} & := K_\delta \cap \{x_2 > 0\}\,; \\
K^{-}_{\delta} & := K_\delta \cap \{x_2 < 0\}\,.
\end{array}
$$

By the symmetry arguments,

$$
\begin{aligned}
\lambda_1^{NND}(\widehat{\mathfrak{D}}(2,\epsilon)) &= \lambda_1(B_{R_2} \setminus K_\delta); \\
\lambda_1^{DND}(\widehat{\mathfrak{D}}(2,\epsilon)) &= \lambda_1(B^{+}_{R_2} \setminus K^{+}_{\delta}).
\end{aligned}
$$

The family of compact sets $(K_\delta)_{\delta>0}$ concentrates to the set $\{x_+, x_-\}$, in the sense that K_δ is contained in any open neighborhood of $\{x_+, x_-\}$ for δ small enough.

Reference [AFHL2018] (Abatangello-Felli-Hillairet-Léna) provides two-term asymptotic expansions in this situation.

$$\lambda_1(B^+_{R_2}\setminus K^+_\delta) = \lambda_1(B^+_{R_2}) + u(x_+)^2 \frac{2\pi}{|\log(\mathrm{diam}(K^+_\delta)|} + o\left(\frac{1}{|\log(\mathrm{diam}(K^+_\delta)|}\right),$$

where $\mathrm{diam}(K^+_\delta)$ is the diameter of K^+_δ and u an eigenfunction associated with $\lambda_1(B^+_{R_2})$, normalized in $L^2(B^+_{R_2})$. Using $\mathrm{diam}(K^+_\delta) = 2R_1 \sin(\delta)$ we get after simplification

$$\lambda_1^{DND}(\widehat{\mathfrak{D}}(2,\epsilon)) = j^2_{1,1} + \\ + \frac{8}{R_2^2}\left(\frac{J_1(j_{1,1}R_1/R_2)}{J_1'(j_{1,1})}\right)^2 \frac{1}{|\log(\pi/2-\epsilon)|} + o\left(\frac{1}{|\log(\pi/2-\epsilon)|}\right), \tag{12}$$

where $j_{\ell,k}$ is the k-th zero of the Bessel function J_ℓ corresponding to the integer $\ell \in \mathbb{N}$.

We obtain a similar expansion for the other eigenvalue, starting from Theorem 1.4 in [AFHL],

$$\lambda_1(B_{R_2} \setminus K_\delta) = \lambda_1(B_{R_2}) + \mathrm{Cap}_{B_{R_2}}(K_\delta, u) + o\left(\mathrm{Cap}_{B_{R_2}}(K_\delta, u)\right).$$

In this formula, u is an eigenfunction associated with $\lambda_1(B_{R_2})$ and normalized in $L^2(B_{R_2})$, and $\mathrm{Cap}_{B_{R_2}}(K_\delta, u)$ is defined in [AFHL].
Since u is radially symmetric, $u(x_+) = u(x_-)$.
We can adapt their proof to give

$$\mathrm{Cap}_{B_{R_2}}(K_\delta, u) = u(x_\pm)^2 \mathrm{Cap}_{B_{R_2}}(K_\delta) + o\left(\mathrm{Cap}_{B_{R_2}}(K_\delta)\right),$$

where $\mathrm{Cap}_{B_{R_2}}(K_\delta)$ is the classical (condenser) capacity of K_δ relative to B_{R_2}.

Since $K_\delta = K_\delta^+ \cup K_\delta^-$, and since K_δ^+ and K_δ^- concentrate to x_+ and x_- respectively, we have

$$\mathrm{Cap}_{B_{R_2}}(K_\delta) \sim \mathrm{Cap}_{B_{R_2}}(K_\delta^+) + \mathrm{Cap}_{B_{R_2}}(K_\delta^-)$$

as $\delta \to 0$. Finally, Proposition 1.6 in [AFHL] gives an asymptotic expansion for $\mathrm{Cap}_{B_{R_2}}(K_\delta^\pm)$.

Gathering these estimates, we find

$$\lambda_1^{NND}(\widehat{\mathfrak{D}}(2,\epsilon)) = j_{0,1}^2 + \\ + \frac{4}{R_2^2}\left(\frac{J_0(j_{0,1}R_1/R_2)}{J_0'(j_{0,1})}\right)^2 \frac{1}{|\log(\pi/2-\epsilon)|} + o\left(\frac{1}{|\log(\pi/2-\epsilon)|}\right). \tag{13}$$

Comparison between (NDD) and (DND) with D-cracks: main proposition

For the two other cases we have weaker results but which are enough for getting, using also the analyticity with respect to ϵ,

Proposition

There exists $\epsilon_0 \in (0, \frac{\pi}{2})$ such that for $\epsilon \in [\epsilon_0, \frac{\pi}{2})$ we have

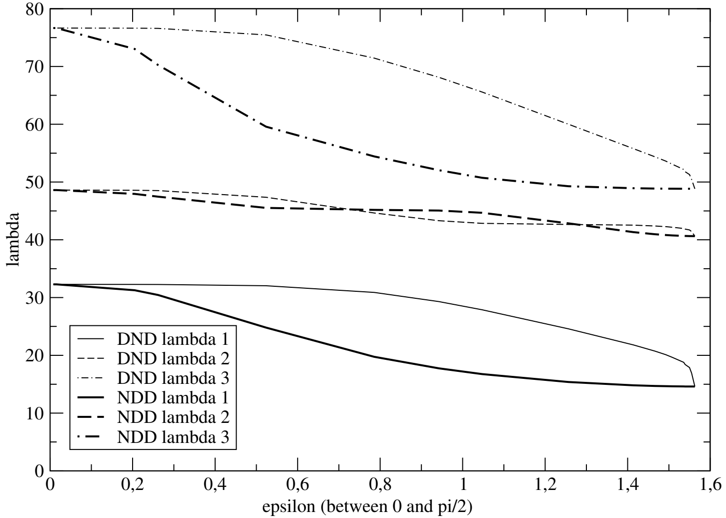
$$\lambda_1^{NDD}(\widehat{\mathfrak{D}}(2,\epsilon)) < \lambda_1^{DND}(\widehat{\mathfrak{D}}(2,\epsilon))\,.$$

Moreover $\delta(\epsilon) := \lambda_1^{NDD}(\widehat{\mathfrak{D}}(2,\epsilon)) - \lambda_1^{DND}(\widehat{\mathfrak{D}}(2,\epsilon))$ can at most vanish in $(0, \frac{\pi}{2})$ on a sequence of ϵ's with no accumulation point except possibly at 0.

A more accurate analysis as $\epsilon \to 0$ would be useful for excluding the possibility of a sequence of zeros of δ tending to 0. Numerics strongly suggests that $\delta(\epsilon)$ is negative in $(0, \frac{\pi}{2})$. The presented arguments are general and not related to $N = 2$.

Comparison between (NDD) and (DND) with D-cracks: numerics

For $\epsilon = 0$ and $\frac{\pi}{2}$ the theory says that the two spectra coincide.The union of these two spectra corresponds to the odd eigenfunctions on $\mathfrak{D}(2, \epsilon)$ which are antisymmetric by inversion.
For the ground state energies, the two curves do not cross and have different curvature properties. This strongly suggests that they are only equal for $\epsilon = 0$ and $\frac{\pi}{2}$. Some crossing (two points) is observed for the curves corresponding to the second eigenvalues. No crossing is observed for the curves corresponding to the third eigenvalues.


80
70
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50
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30
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lambda
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epsilon (between 0 and pi/2)
DND lambda 1
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NDD lambda 3

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