Eigenvalue asymptotics of large Toeplitz matrices with random perturbations

Johannes Sjöstrand
IMB, Université de Bourgogne

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1. Introduction and background

For self-adjoint (pseudo-)differential operators with discrete spectrum we know that the eigenvalues are distributed according to the Weyl law, under quite general assumptions. This holds for large eigenvalues and also in the semi-classical limit: Let $P = P(x, hD; h)$ with $P(x, \xi; h) = p(x, \xi) + \mathcal{O}(h)$. Then

$$\#(\sigma(P) \cap \Omega) = (2\pi h)^{-n} \left(\mathrm{vol}\left(p^{-1}(\Omega)\right) + o(1)\right), \; 0 < h \to 0. \quad (1)$$

vol $=$ the volume in real phase space, $\sigma(P) =$ the spectrum of P; we only consider operators with discrete spectrum.

Non-self-adjoint (pseudo-)differential operators. Here, the situation is more complicated. When the coefficients are analytic, the spectrum may depend on the behaviour of the symbol in the complex domain (e.g. by the complex WKB method), while the natural Weyl law refers only to the behaviour in real phase space, and (1) may fail to hold.

Another feature of general non-self-adjoint operators is the spectral instability; the resolvent may be very large away from $\sigma(P)$:

$$\|(z-P)^{-1}\| \gg \frac{1}{\mathrm{dist}\,(z,\sigma(P))}.$$

In the case of differential operators this follows from quasi-mode constructions for $P-z$ or its adjoint: Hörmander [Ho60a, Ho60b], E.B. Davies [Da99], M. Zworski [Zw01], N. Dencker–Sj–Zworski [DeSjZw04].
Equivalently, the spectrum can be very unstable under small perturbations of the operator.

It is natural to add a small random perturbation. One line of research concerns the case of elliptic (pseudo-)differential operators with small random perturbations, M. Hager, W. Bordeaux Montrieux, Sj, Vogel, 2005–present: Under quite general assumptions, we have Weyl asymptotics with probability close to 1 for the distribution of eigenvalues:

$$\#(\Omega \cap \sigma(P + \delta Q)) \approx \left(\frac{2\pi}{h}\right)^n \mathrm{vol}\, \left(p^{-1}(\Omega)\right),$$

where $P = P(x, hD; h)$ with leading semi-classical symbol $p(x, \xi)$, $\Omega \subset \mathbf{C}$. See [SjBook19]. Q can be the operator of multiplication with a random linear combination of eigenfunctions of an auxiliary self-adjoint differential operator.

In this talk we will not discuss resonances; there have been some works with random perturbations and plenty open problems remain.

Example.

Consider a finite difference operator on S^1

$$P = \sin(hD) \circ (1 + ae^{ix} + a_{\mathrm{m}} e^{-ix} + be^{2ix} + b_{\mathrm{m}} e^{-2ix}) \circ \sin(hD) + ce^{ix} + c_{\mathrm{m}} e^{-ix}, \tag{2}$$

$$p = (1 + ae^{ix} + a_{\mathrm{m}} e^{-ix} + be^{2ix} + b_{\mathrm{m}} e^{-2ix})(\sin \xi)^2 + ce^{ix} + c_{\mathrm{m}} e^{-ix}, \tag{3}$$

for suitable values of the coefficients.

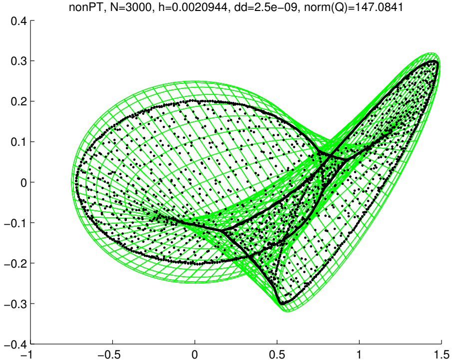


Figure: Eigenvalues of $P_\delta = P + \delta Q$ as in (2), $h = 0.0020944$, $N = 3000$, $\delta = 2.5 \times 10^{-9}$. $\|Q\| = 147.0841$.

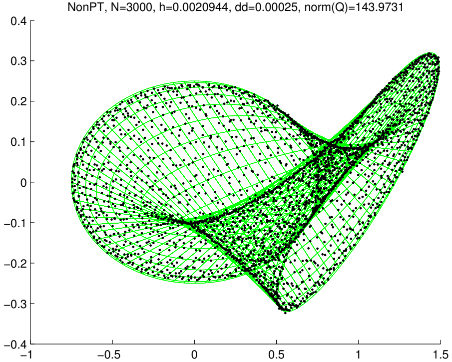


Figure: Eigenvalues of $P_\delta = P + \delta Q$ as in (2) ff., $h = 0.0020944$, $N = 3000$, $\|Q\| = 143.9731$. $\delta = 2.5 \times 10^{-4}$.

In this talk we discuss large Toeplitz matrices with random perturbations, an interesting example of boundary value problems. The first example is that of a large Jordan block:

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & \dots & . \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} : \mathbf{C}^N \to \mathbf{C}^N.$$

- $D(0,1)$ (the open unit disc) is a region of spectral instability ([Zw02]).
- We have spectral stability (a good resolvent estimate) in $\mathbf{C} \setminus \overline{D(0,1)}$.
- $\sigma(A_0) = \{0\}$.

Thus, if $A_\delta = A_0 + \delta Q$ is a small (random) perturbation of A_0 we expect the eigenvalues to move inside a small neighborhood of $\overline{D(0,1)}$.

In the special case when $Qu = (u|e_1)e_N$, where $(e_j)_1^N$ is the canonical basis in $\mathbf{C}^N$, the eigenvalues of A_δ are of the form

$$\delta^{1/N}e^{2\pi ik/N},\ k \in \mathbf{Z}/N\mathbf{Z},$$

so if we fix $0 < \delta \ll 1$ and let $N \to \infty$, the spectrum "will converge to a uniform distribution on S^1".

Davies and M. Hager [DaHa09] considered random perturbations $\delta Q = \delta(q_{j,k}(\omega))$, where typically $q_{j,k} \sim \mathcal{N}_{\mathbf{C}}(0,1)$, independent. They showed under quite general assumptions that with probability close to one, most of the eigenvalues are close to the circle of radius $\delta^{1/N}$.

The angular distribution was not treated in [DaHa09]. In [SjBook19], I did so, using the general methods that have been developed for the case of elliptic PDE [Ha05],... . A. Guionnet, P. Matchett Wood and O. Zeitouni [GuMaZe14] studied the convergence of the counting measure. Both results show that the eigenvalues have a tendency of accumulating uniformly along the unit circle (when δ is neither too small nor too big). This is another example of Weyl asymptotics, associated to the symbol

$$p(\xi) = e^{i\xi} \text{ on } S^1_\xi .$$

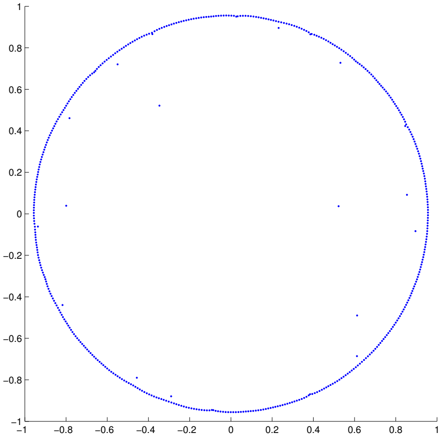
Expected eigenvalue density inside. With Vogel [SjVo14], we have investigated the expected eigenvalue distribution inside $D(0,1)$ in cases when $\delta^{1/N} \approx 1$, by adapting the methods of [Vo14] related to classical works on zeros of random polynomials by M. Kac, B. Shiffmann–S. Zelditch, cf. [HoKrPeVi09].
We showed roughly that with the same random perturbation, the expected density of eigenvalues inside the unit disc is given up to small errors by

$$\frac{1}{2\pi}\frac{4}{(1-|z|^2)^2}L(dz) + \text{a small remainder}, \;\; L(dz) = d\Re z \, d\Im z.$$

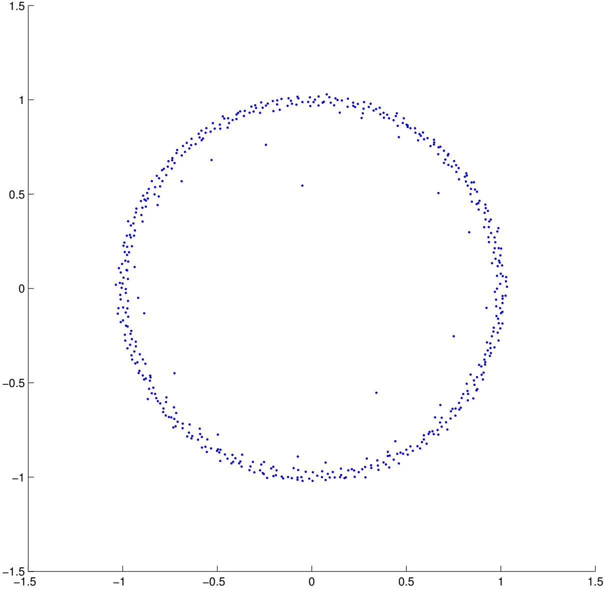
There is a general theoretical study of eigenvalue density away from the main accumulation; C. Bordenave, M. Capitaine [BoCa16].

Numerical illustrations in the Jordan case.

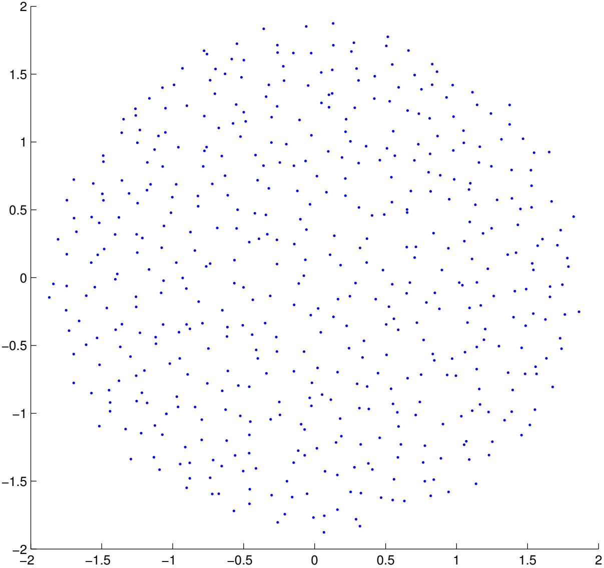
$N = 500$, $\delta/\sqrt{2}$ ranges from 10^{-11} to 0.02. First $\delta/\sqrt{2} = 10^{-11}$:



$\delta/\sqrt{2} = 0.0075$



$\delta/\sqrt{2} = 0.03$



Bidiagonal Toeplitz matrices.

Consider the bidiagonal $N \times N$ matrices:

$$P = P_{\mathrm{I}} = \begin{pmatrix} 0 & a & 0 & .. & .. & 0 \\ b & 0 & a & .. & .. & 0 \\ 0 & b & 0 & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & .. & 0 & a \\ 0 & 0 & .. & .. & b & 0 \end{pmatrix} \tag{4}$$

$$P = P_{\mathrm{II}} = \begin{pmatrix} 0 & a & b & 0 & .. & .. & 0 \\ 0 & 0 & a & b & .. & .. & 0 \\ 0 & 0 & 0 & a & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & a & b \\ .. & .. & .. & .. & .. & 0 & a \\ .. & .. & .. & .. & .. & 0 & 0 \end{pmatrix}, \tag{5}$$

for $N \gg 1$, where $a,\, b \in \mathbf{C} \setminus \{0\}$.

Identifying $\mathbf{C}^N \simeq \ell^2([1,N]) \simeq \ell^2_{[1,N]}(\mathbf{Z}) =$ the space of all $u \in \ell^2(\mathbf{Z})$ with support in $[1,N]$), $[1,N] = \{1,2,...,N\}$, we have:

$$P_{\mathrm{I}} = 1_{[1,N]}(a\tau_{-1} + b\tau_1) = 1_{[1,N]}(ae^{iD_x} + be^{-iD_x}), \qquad (6)$$

$$P_{\mathrm{II}} = 1_{[1,N]}(a\tau_{-1} + b\tau_{-2}) = 1_{[1,N]}(ae^{iD_x} + be^{2iD_x}), \qquad (7)$$

where $\tau_k = \tau^k$ denotes translation by k, $\tau = \tau_1$.
The symbols of these operators are by definition,

$$P_{\mathrm{I}}(\xi) = ae^{i\xi} + be^{-i\xi},\ \ P_{\mathrm{II}}(\xi) = ae^{i\xi} + be^{2i\xi}. \qquad (8)$$

$E_1 := P_{\mathrm{I}}(S^1)$, $S^1 \simeq \mathbf{R}/2\pi\mathbf{Z}$ is equal to the ellipse with focal points $\pm 2\sqrt{ab}$ and semi-axes of length $|a|+|b|$ and $||a|-|b||$.

- The spectrum $\sigma(P_{\mathrm{I}})$ of the operator P_{I} can be computed explicitly and is contained in the focal segment $[-2\sqrt{ab}, 2\sqrt{ab}]$.
- The numerical range is contained in the convex hull of E_1.

We consider the following random perturbation of $P_0 = P_{\mathrm{I}}$

$$P_\delta := P_0 + \delta Q_\omega, \quad Q_\omega = (q_{j,k}(\omega))_{1 \le j,k \le N}, \tag{9}$$

where $0 \le \delta \ll 1$ and $q_{j,k}(\omega)$ are independent and identically distributed complex Gaussian random variables $\sim \mathcal{N}_{\mathbf{C}}(0, 1)$.

In [SjVo15] with Vogel, we considered $P_0 = P_{\mathrm{I}}$, where $a, b \in \mathbf{C}$ satisfy $0 < |b| < |a|$. Let P_δ be as in (9). For δ in a reasonable parameter range we showed that most of the eigenvalues are close to the ellipse E_1 and their distribution satisfies the natural Weyl law. In [SjVo16] we also considered the expected density inside and got an expression which is less explicit than in the Jordan case.

Numerical illustrations in the bidiagonal case.

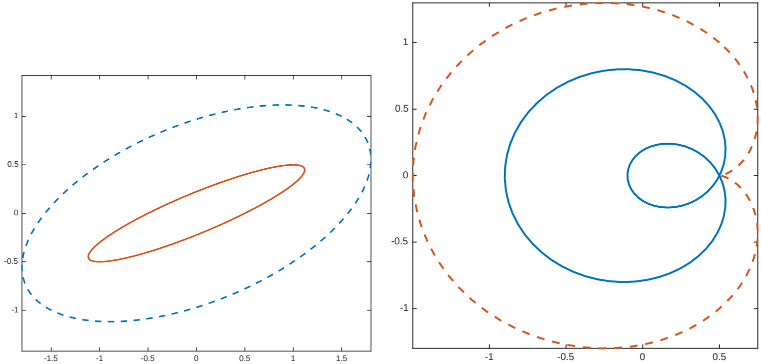


Figure: The left hand side shows the image of S^1 under the principal symbol of case I (for the dashed ellipse we chose $b = 0.5$, $a = 1 + i$ and for the other ellipse $b = 0.5$, $a = 0.5 + 0.5i$). The right hand side is similar but for the principal symbol of case II (for the dashed line we chose $b = 0.5$, $a = i$ and for the continuous line $b = 0.5$, $a = 0.4i$).

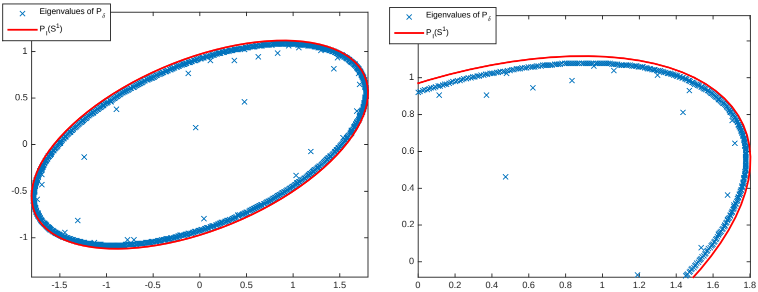


Figure: The spectrum of P_I with $N = 500$, $a = 1 + i$ and $b = 0.5$ perturbed with a complex Gaussian random Matrix with coupling constant $\delta = 10^{-5}$. The red line is the image of the unit circle S^1 under the symbol P_I.

2. Recent results

Let $N_\pm \in \mathbf{N} = \{0, 1, ...\}$ with $N_+ + N_- \neq 0$. Let $a_j \in \mathbf{C}$, $-N_- \leq N_+$ with $a_{\pm N_\pm} \neq 0$ when $N_\pm = 0$. Consider the operator

$$p(\tau) := \sum_{-N_-}^{N_+} a_j \tau^j, \tag{10}$$

acting on functions on $\mathbf{Z}$ or on $\mathbf{Z}/M\mathbf{Z}$. The symbol of $\tau = e^{-iD_x}$ is equal to $1/\zeta$, with $\zeta = e^{i\xi}$ and the symbol of $p(\tau)$ is given by the meromorphic function

$$\mathbf{C} \ni \zeta \mapsto p(1/\zeta) = \sum_{-N_-}^{N_+} a_j \zeta^{-j}. \tag{11}$$

We get the $N \times N$ Toeplitz matrix,

$$P_N := 1_{[1,N]} p(\tau) 1_{[1,N]}, \tag{12}$$

acting on $\mathbf{C}^N \simeq \ell^2([1,N]) \simeq \ell^2_{[1,N]}(\mathbf{Z})$. For

$$A_0, \ P_{\mathrm{I}} \text{ and } P_{\mathrm{II}},$$

$p(1/\zeta)$ is equal to

$$\zeta, \ a\zeta + b/\zeta \text{ and } a\zeta + b\zeta^2$$

respectively. Similarly, we write $P_{\mathbf{N}} = 1_{\mathbf{N}} p(\tau) 1_{\mathbf{N}}$ and let $P_{\mathbf{Z}}$ and $P_{\mathbf{Z}/M\mathbf{Z}}$ denote $p(\tau)$, acting on functions on $\mathbf{Z}$ and $\mathbf{Z}/M\mathbf{Z}$ respectively. By Fourier series, we have

$$\sigma(P_{\mathbf{Z}}) = p(S^1), \ \sigma(P_{\mathbf{Z}/N\mathbf{Z}}) = p(S^1_N),$$

where $S^1_N = \{e^{2\pi i k/N}; \, 1 \leq k \leq N\}$. We can view P_N as a uniformly finite rank perturbation of $P_{\mathbf{Z}/N\mathbf{Z}}$.

Let $P_N^\delta = P_N + \delta Q$, where $Q = Q_\omega = (q_{j,k}(\omega))_{1\le j,k\le N}$ and $q_{j,k}(\omega) \sim \mathcal{N}_{\mathbf{C}}(0,1)$ are independent complex Gaussian random variables with expectation 0 and variance 1. We consider the case $N \gg 1$, $0 \le \delta \ll 1$.

Counting eigenvalues in a fixed domain.

Let $\Omega \Subset \mathbf{C}$ be an open simply connected set with smooth boundary $\partial\Omega$ which is independent of N. We suppose that

(Ω1) $\partial\Omega$ intersects $p(S^1)$ in at most finitely many points;

(Ω2) the points of intersection are non-degenerate, i.e.

$$\partial_\zeta p \neq 0 \text{ on } p^{-1}(\partial\Omega \cap p(S^1)); \tag{13}$$

(Ω3) $\partial\Omega$ intersects $p(S^1)$ transversally, in the following sense: for each $z_0 \in \partial\Omega \cap p(S^1)$ let $\gamma_k \subset p(S^1)$, $k = 1, \dots, n$ denote the mutually distinct segments of $p(S^1)$ passing through z_0, i.e. each γ_k is given by the image of a small neighborhood in S^1 of a point in $p^{-1}(z_0) \cap S^1$. Then γ_k and $\partial\Omega$ intersect transversally at z_0.

Theorem ([SjVo19a])

Let p, P_N^δ, $M = N_+ + N_-$ *and* Ω *be as above. Let* Ω *be as above, satisfying conditions* $(\Omega 1)$–$(\Omega 3)$ *and pick a* $\delta_0 \in]0,1[$. *There exists a constant* $C > 0$, *such that, for* $N > 1$ *sufficiently large, if*

$$Ce^{-N^{\delta_0}/(2M)} \leq \delta \leq \frac{N^{-4}}{C}, \tag{14}$$

then

$$\left| \#(\sigma(P_N^\delta) \cap \Omega) - \frac{N}{2\pi} \int_{p^{-1}(\Omega)\cap S^1} L_{S^1}(d\theta) \right| \leq \mathcal{O}(N^{\delta_0} \log N). \tag{15}$$

with probability

$$\geq 1 - \mathcal{O}(\log N) \left(e^{-N^2} + \delta^{-M} e^{-\frac{1}{2} N^{\delta_0}} \right). \tag{16}$$

We have a related result on the a.s. weak convergence of the counting measure $\xi_N = N^{-1} \sum_{\lambda \in \sigma(P_N^\delta)} \delta_\lambda$, for the eigenvalues, very closely related to general results of the same type by A. Basak, E. Paquette, O. Zeitouni [BaPaZe18].

Corollary ([SjVo19a])

Let $\delta_0 \in]0,1[$, and write $M = N_+ + N_-$. Then, there exists a constant $C > 0$ such that if $\delta = \delta_N$ satisfies (14),

$$Ce^{-N^{\delta_0}/(2M)} \leq \delta \leq \frac{N^{-4}}{C},$$

then, almost surely,

$$\xi_N \rightharpoonup p_* \left(\frac{1}{2\pi} L_{S^1} \right), \quad N \to \infty, \tag{17}$$

weakly, where L_{S^1} denotes the Lebesgue measure on S^1.

We also have a result about counting eigenvalues in thin neighborhoods of $p(S^1)$.

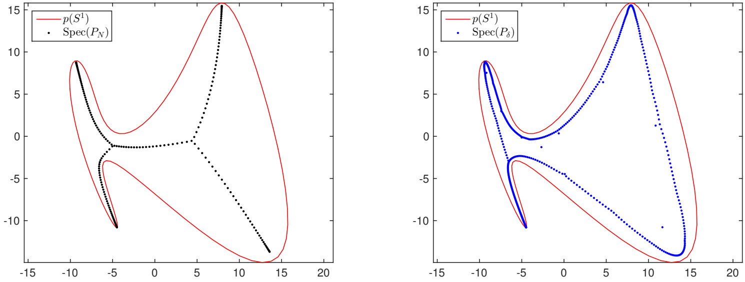


Figure: The pictures on the left hand side shows the spectrum of the Toeplitz matrix P_N given by the symbol $p(1/\zeta) = 2i\zeta^{-1} + \zeta^2 + \frac{7}{10}\zeta^3$ and the right hand side shows the spectrum of P_δ, with $\delta = 10^{-12}$. The red line shows the symbol curve $p(S^1)$. $N_{\mathrm{left}} = 160$, $N_{\mathrm{right}} = 500$

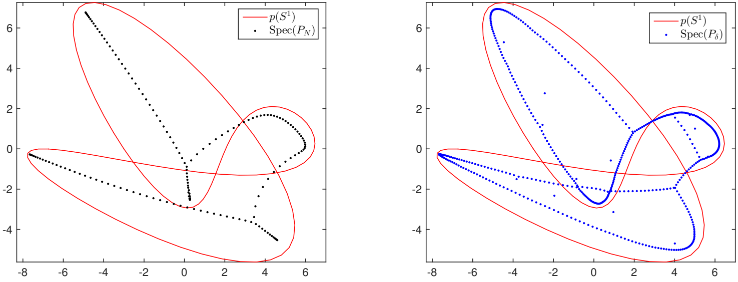


Figure: The pictures on the left hand side shows the spectrum of the Toeplitz matrix P_N given by the symbol $p(1/\zeta) = 2\zeta^{-3} - \zeta^{-2} + 2i\zeta^{-1} - 4\zeta^2 - 2i\zeta^3$ and the right hand side shows the spectrum of a random perturbation P_δ, with coupling constant $\delta = 10^{-12}$. The red line shows the symbol curve $p(S^1)$.

Recently Basak and Zeitouni [BaZe19] have given quite a technical description of the expected eigenvalue density away from $p(S^1)$ for general "finite band" Toeplitz matrices. As their examples they recover (basically) the results of [SjVo14], [SjVo16].

An extension to less regular symbols

Now consider

$$p(\tau) = \sum_{-\infty}^{+\infty} a_\nu \tau^\nu,$$

where

$$\sum |\nu a_\nu| < \infty$$

Let

$$P_N = 1_{[0,N[} p(\tau) 1_{[0,N[} : \ \ell^2([0,N[) \to \ell^2([0,N[).$$

We have the symbol $p(1/\zeta) \in C^1(S^1)$. $\gamma = p(S^1)$ is a C^1-curve. Let Ω be domain in $\mathbf{C}$ as above, satisfying (Ω1)–(Ω3). For simplicity we add the assumption

(Ω4) γ has no point of self intersection at $\gamma \cap \partial\Omega$.

Theorem ([SjVo19b])

Let $\delta_0 \in]0,1[$, $\delta_1 > 3/2$. *If* $e^{-N^{\delta_0}} \le \delta \ll N^{-\delta_1}$, *then* $\exists\ \epsilon_N = o(1)$, $N \to \infty$, *such that*

$$\left| \#(\sigma(P_N^\delta) \cap \Omega) - \frac{N}{2\pi} \int_{p^{-1}(\Omega) \cap S^1} L_{S^1}(d\theta) \right| \le \epsilon_N N, \tag{18}$$

with probability

$$\ge 1 - e^{-N^{\delta_0}}. \tag{19}$$

We have the corresponding corollary about the weak convergence of the counting measure.

R. Movassagh has informed us about his work with L.P. Kadanov [MoKa16] about Toeplitz operators with singular symbols. This seems to open new horizons.

3. Some elements of the proofs

As in [SjVo14, SjVo15, SjVo16] we make a Grushin (Feschbach) reduction to a matrix of fixed size independent of N. In the cited works, the reduction depends very much on the type of the operator. We now have a unified approach.

Let $J = [-N_-, N_+[$ be an interval in $\mathbf{Z}$ of length $N_+ + N_- =: M$. Then $\mathbf{Z}/(N+M)\mathbf{Z} = J \cup I_N$, where I_N is an "interval" of "length" $\# I_N = N$. We have $P_N \simeq P_{I_N}$. Identify $\ell^2(S_{N+N_++N_-}) \simeq \ell^2(I_N) \oplus \ell^2(J)$. Then

$$P_{\mathbf{Z}/(N+M)\mathbf{Z}} - z = \mathcal{P}_N(z) = \begin{pmatrix} P_{I_N} - z & R_-^N \\ R_+^N & R_{+-}(z) \end{pmatrix} \qquad (20)$$

where

$$P_{I_N} - z = 1_{I_N}(p(\tau) - z)1_{I_N},\ R_-^N = 1_{I_N} p(\tau) 1_J,$$
$$R_+^N = 1_J p(\tau) 1_{I_N},\ R_{+-}^N = 1_J(p(\tau) - z)1_J.$$

$P_{\mathbf{Z}/(N+M)\mathbf{Z}}$ is normal with spectrum equal to $p(S^1_{N+M})$, so if z is outside that set, $\mathcal{P}_N(z)$ is invertible with inverse

$$\begin{pmatrix} E^N(z) & E^N_+(z) \\ E^N_-(z) & E^N_{-+}(z) \end{pmatrix}, \quad E^N_{-+}(z) = 1_J(P_{\mathbf{Z}/(N+M)\mathbf{Z}} - z)^{-1} 1_J.$$

Define $\mathcal{P}^\delta_N(z)$ by replacing P_N with P^δ_N in (20). It is bijective for $z \notin p(S^1_{N+M})$ when δ is small enough. Let $E^\delta_{-+}(z)$ be the lower left entry of the inverse. We can study the determinant via

$$\ln|\det(P^\delta_N - z)| = \ln|\det \mathcal{P}^\delta_N(z)| + \ln|\det E^\delta_{-+}(z)|,$$

and use that $E^\delta_{-+}(z)$ has a perturbative expansion:

$$E^\delta_{-+}(z) = E_{-+}(z) - \delta E_- Q E_+ + "\mathcal{O}(\delta^2)".$$

We also need

Lemma

The M largest singular values of E_+ and of E_- are bounded from below by $1/\mathrm{Const.}$, uniformly in N for $N \gg M$.

The proof is elementary and uses in an essential way that $p(\tau)$ in (10) is a finite difference operator with $a_{N_+}, a_{-N_-} \neq 0$.

In the proof of Theorem 3.1, we construct a similar Grushin problem with a sufficiently large interval J. The lemma will not necessarily hold, but we can formulate a second Grushin problem for E_{-+}, related to its small singular values. The two Grushin problems can be composed in the natural way and gives a new one for which the lemma holds.

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Thank you for your attention.

Bon anniversaire André!

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