Gevrey estimates of the resolvent in N-body problems

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Introduction

Motivation : Sub-exponential time-decay of quantum dynamics.

Study the Fokker-Planck operator P by scattering method:

$$P = v \cdot \nabla_x - \nabla W(x) \cdot \nabla_v - \Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}$$

with $W \in C^1(\mathbb{R}^n; \mathbb{R})$. The Maxwell distribution

$$\mathfrak{m}(x, v) = (2\pi)^{\frac{n}{4}} e^{-\frac{1}{2}(\frac{v^2}{2} + W(x))}$$

always verifies $P\mathfrak{m} = 0$. If $n = 3$ and $W(x)$ decreases quickly, then (W, 2015)

$$e^{-tP} = \frac{1}{4\pi t^{\frac{3}{2}}}(\langle \mathfrak{m}, \cdot \rangle \mathfrak{m} + O(t^{-\epsilon})), \quad t \to +\infty. \tag{1}$$

Motivation

Question raised in [W., 2015]. Assume $W(x) \simeq \langle x \rangle^{\rho}$ for some $0 < \rho < 1$. Can one prove the return to equilibrium

$$e^{-tP} = \langle \mathfrak{m}, \cdot \rangle \mathfrak{m} + O(e^{-at^{\frac{\rho}{2-\rho}}}) \quad \text{as} \quad t \to +\infty?$$

Here $W(x)$ is normalized by $\int_{\mathbb{R}^n} e^{-W(x)} dx = 1$. (The normalization fr $W(x)$ used in (1) is: $W(x) \to 0$ as $|x| \to \infty$).

An affirmative answer is obtained by T. Li and Z. Zhang (Sci. China Math. (2018)).

Motivation

The associated Witten Laplacian is given by

$$-\Delta_W = -\Delta + U(x), \quad U(x) = \frac{1}{4}|\nabla W(x)|^2 - \frac{1}{2}\Delta W(x).$$

If $W(x) \sim \langle x \rangle^\rho$ for some $0 < \rho < 1$ and for $|x|$ large, then $U(x) \geq \frac{c}{|x|^{2(1-\rho)}}$ for $|x|$ large. In this case, $-\Delta_W$ is a compactly supported perturbation of a model operator $-\Delta + V_0(x)$ where $V_0(x) \geq \frac{c}{\langle x \rangle^{2\mu}}$ for all $x \in \mathbb{R}^n$, where $c > 0$ and $\mu = 1 - \rho \in]0, 1[$ and zero is an eigenvalue of $-\Delta_W$ embedded in its essential spectrum. Sometimes, it may be necessary to include other terms which are non-selfadjoint.

This leads us to study a class of non-selfadjoint operators with zero as embedded eigenvalue.

Schrödinger operators with positive potential

If $H = -\Delta + V(x)$ where $V(x)$ is real and

$$\frac{c}{\langle x\rangle^{2\mu}} \leq V(x) \leq \frac{C}{\langle x\rangle^{2\mu}}, \quad \mu \in]0,1[, C, c > 0,$$

then

- D. Yafaev (1982): if $n = 1$ and V is analytic, then $\forall \chi \in C_0^\infty(\mathbb{R})$,

$$\chi(x) e^{-itH} \chi(x) = O(e^{-a|t|^{\frac{1-\mu}{1+\mu}}}), \quad t \in \mathbb{R}.$$

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- S. Nakamura (1994): $n \ge 1$, $\forall \chi \in C_0^\infty(\mathbb{R}^n)$,

$$\chi(\frac{|x|}{t^{\frac{1}{1+\mu}}})e^{-tH} = O(e^{-at^{\frac{1-\mu}{1+\mu}}}), \quad t > 0.$$

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- Method of Markov processes: P. Cattiaux, A. Guillin, $\cdots$.

The model operator

We first recall some known results on Gevrey estimate of the resolvent. Let H_0 be an operator of the form

$$H_0 = -\sum_{i,j=1}^{n} \partial_{x_i} a^{ij}(x) \partial_{x_j} + \sum_{j=1}^{n} b_j(x) \partial_{x_j} + V(x), \tag{2}$$

where $a^{ij}(x)$, $b_j(x)$ and $V(x)$ are complex-valued functions. Suppose that $a^{ij}, b_j \in C_b^1$ and that there exists $c > 0$ such that

$$\text{Re}\,(a^{ij}(x)) \geq c I_n, \quad \forall x \in \mathbb{R}^n. \tag{3}$$

Assume that V is relatively bounded w.r.t. $-\Delta$ with relative bound zero.

The model operator

Assume that there exists some constants $0 < \mu < 1$ and $c_0 > 0$ such that

$$|\langle H_0 u, u\rangle| \geq c_0(\|\nabla u\|^2 + \|\langle x\rangle^{-\mu} u\|^2), \quad \text{for all } u \in H^2, \tag{4}$$

$$\sup_x |\langle x\rangle^{\mu} b_j(x)| < \infty, \quad j = 1, \cdots, n. \tag{5}$$

(4) is called weighted coercive condition. Assume in addition Re $H_0 \geq 0$. Then (4) is satisfied by $H_0 - \lambda$ for all $\lambda \leq 0$ with the same constant c_0.

A uniform energy estimate

For $s \in \mathbb{R}$, denote

$$\varphi_s(x) = (1 + \frac{|x|^2}{R_s^2})^{s/2},$$

where $R_s = M\langle s\rangle^{\frac{1}{1-\mu}}$ for some $M > 1$ independent of s.

Lemma 1

Under the above conditions on H_0, there exists some $C, M > 0$ such that

$$\|\langle x\rangle^{-\mu}\varphi_s(x)u\| + \|\nabla(\varphi_s(x)u)\| \leq C\|\langle x\rangle^{\mu}\varphi_s(x)(H_0 - \lambda)u\| \qquad (6)$$

for any $s \in \mathbb{R}$, $\lambda \leq 0$ and $u \in \{f \in H^2(\mathbb{R}^n); \langle x\rangle^{|s|+\mu} f \in L^2\}$.

Consequence. Let $R_0(\lambda) = (H_0 - \lambda)^{-1}$ for $\lambda \notin \sigma(H_0)$. Then

$$\|\langle x\rangle^{-\mu}\varphi_s R_0(\lambda)\varphi_{-s}\langle x\rangle^{-\mu}\| + \|\nabla(\varphi_s R_0(\lambda)\varphi_{-s})\langle x\rangle^{-\mu}\| \leq C$$

uniformly in $\lambda < 0$ and $s \in \mathbb{R}$.

Gevrey estimates of the resolvent

Under the assumptions (4) and (5), H_0 is bijective from $D(H_0)$ to $R(H_0)$. Let $R_0(0) : R(H_0) \to D(H_0)$ be its algebraic inverse. Denote

$$L^{2,s} = L^2(\mathbb{R}^n_x; \langle x\rangle^{2s} dx).$$

$R_0(0)$ is a densely defined and closed operator, continuous from $L^{2,s}$ to $L^{2,s-2\mu}$ for any $s \in \mathbb{R}$. Thus $R_0(0)^N : L^{2,s} \to L^{2,s-2\mu N}$ is well defined for any $s \in \mathbb{R}$. One has

$$s - \lim_{z\in\Omega(\delta), z\to 0} \langle x\rangle^{-2N\mu}(R_0(z)^N - R_0(0)^N) = 0,$$

where $\Omega(\delta) = \{z; |\arg z| > \frac{\pi}{2} + \delta\}, \delta > 0.$

Gevrey estimates of the model resolvent

From Lemma 1, we deduce that

$$\|\langle x\rangle^{-\mu}\varphi_s R_0(\lambda)\varphi_{-s}\langle x\rangle^{-\mu}\| + \|\nabla(\varphi_s R_0(\lambda)\varphi_{-s})\langle x\rangle^{-\mu}\| \leq C \qquad (7)$$

uniformly in $\lambda \leq 0$ and $s \in \mathbb{R}$.

Theorem 2

For any $a > 0$, there exists $C_a > 0$ such that

$$\|e^{-a\langle x\rangle^{1-\mu}} R_0(\lambda)^N\| + \|R_0(\lambda)^N e^{-a\langle x\rangle^{1-\mu}}\| \leq C_a^{N+1} N^{\gamma N}, \forall N \geq 1.$$

uniformly in $\lambda \leq 0$. Here $\gamma = \frac{2\mu}{1-\mu}$.

Gevrey estimates of the model resolvent

From Theorem 2, one deduces that for any $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi R_0(z)$ belongs to the Gevrey class $\mathcal{G}^{(1+\gamma)}(\Omega(\delta))$:

$$\|\chi(x) R_0^{(N)}(z)\| \le C_\chi C^N N! N^{\gamma N},$$

uniformly in $z \in \Omega(\delta)$ and $N \in \mathbb{N}$. Here $R_0^{(N)}(z) = \frac{d^N}{dz^N} R_0(z)$, $C > 0$ is independent of χ and $1 + \gamma = \frac{1+\mu}{1-\mu}$. If H_0 is in addition selfadjoint, one can take $\Omega(\delta)$ as

$$\Omega(\delta) = \{z; |\arg z| > \delta\}.$$

Quantum Coulomb Hamiltonian

Example. Consider Coulomb Hamiltonian $H_0 = -\Delta + \frac{c}{|x|}$ in $L^2(\mathbb{R}^3)$, $c \in \mathbb{R}$. Let $\chi \in C_0^\infty(\mathbb{R}^3)$. If $c > 0$, then $H_0 \geq 0$ and one has

$$\chi(H_0 - z)^{-1}\chi \in C_b^\infty(\Omega_1)$$

where $\Omega_1 = \{|z| < 1; z \notin \mathbb{R}_+\}$ (S. Nakamura, 1994).

If $c < 0$, one has

$$\chi(H_0 - z)^{-1}\chi \in C_b^\infty(\Omega_\pm)$$

where $\Omega_\pm = \{|z| < 1, 0 < \pm \arg z < \pi - \delta\}$ (S. Fournais and E. Skibsted (2004)).

Gevrey-3 estimates

If $c > 0$, H_0 satisfies the weighted coercive condition with $\mu = \frac{1}{2}$. Therefore

$$R_0(z) \in \mathcal{G}^{(3)}(\Omega(\delta))$$

In addition, the dilated operator

$$H_0(\theta) = e^{-2\theta}\Delta + \frac{ce^{-\theta}}{|x|}$$

still satisfies the weighted coercive condition if $|\theta|$ small and $\operatorname{Im}\theta > 0$. We deduce from Theorem 2 and the numerical range of $H_0(\theta)$ that for $\operatorname{Im}\theta > 0$, $R_0(z,\theta) = (H_0(\theta) - z)^{-1} \in \mathcal{G}^{(3)}(\Omega_2)$,

$$\Omega_2 = \{z \in \mathbb{C}; -c_0 < \arg z < \frac{3\pi}{2}\}$$

for some $c_0 = c_0(\operatorname{Im}\theta) > 0$.

Gevrey-3 estimates

If $c < 0$, Theorem 2 does not apply to H_0, but applies to the dilated operator $H_0(\theta) = e^{-2\theta}\Delta + \frac{ce^{-\theta}}{|x|}$, as soon as Im $\theta > 0$, because for $\theta = i\tau$

$$|\langle u, H_0(i\tau)u\rangle| = \left\{ \|\nabla u\|^4 + 2c\cos\tau \|\nabla u\|^2 \cdot \|\frac{u}{|x|^{\frac{1}{2}}}\|^2 + c^2 \|\frac{u}{|x|^{\frac{1}{2}}}\|^4 \right\}^{\frac{1}{2}}.$$

We deduce from Theorem 2 that for Im $\theta > 0$,

$$R_0(z,\theta) = (H_0(\theta) - z)^{-1} \in \mathcal{G}^{(3)}(\Omega_3)$$

where $\Omega_3 = \{z \in \mathbb{C}; -c_0 < \arg z < \pi - \delta\}$.

Gevrey-3 estimates

By using analytic distortion outside support of χ and comparing $\chi R_0(z)\chi$ with $\chi R_0(z,\theta)\chi$, one can prove

Proposition 3

Let $n = 3$, $H_0 = -\Delta + \frac{c}{|x|}$ with $c \in \mathbb{R}^*$ and $\chi \in C_0^\infty(\mathbb{R}^3)$. If $c > 0$, one has

$$\chi R_0(z)\chi \in \mathcal{G}^{(3)}(\Omega_2)$$

and if $c < 0$ one has

$$\chi R_0(z)\chi \in \mathcal{G}^{(3)}(\Omega_3).$$

Perturbed operator

Let H_0 verify Conditions of Theorem 2. To study quantum resonances near 0, we need some additional conditions. To be simple, consider the case $n = 3$ and $H_0 = -\Delta + V_0(x)$ with

$$V_0(x) = \frac{a - ib}{\langle x \rangle^{2\mu}} \quad \text{or} \quad \frac{a - ib}{|x|},$$

with $\mu \in]0, 1[$, $a, b \geq 0$ with $a + b > 0$. We can also take $V_0(x) = V_1(x) - iV_2(x)$ with $V_2(x) \geq c\langle x \rangle^{-2\mu}$ with V_j both dilation and distortion analytic. Let

$$H = -\Delta + V(x) = H_0 + W(x)$$

a compactly supported perturbation of H_0: $W = V - V_0 \in L^\infty_{\text{comp}}(\mathbb{R}^3; \mathbb{C})$. We want to study large-time expansion of e^{-itH} as $t \to +\infty$.

Positive resonances

Definition 4

A number $\lambda > 0$ is called outgoing resonance of $H = -\Delta + V$ if -1 is an eigenvalue of the compact operator $R_0(\lambda + i0)W$ in $L^{2,-s}$, $s > \frac{1}{2}$. Denote $r_+(-\Delta + V)$ the set of outgoing resonances of H. For $\lambda \in r_+(H)$, define $m_+(\lambda)$ as the algebraic multiplicity of eigenvalue -1 of the compact operator $R_0(\lambda + i0)W$. Similarly, one can define incoming positive resonances.

If V is of short-range, one can show that $\lambda > 0$ is a positive resonance of H if and only if $(H - \lambda)u = 0$ admits a non-trivial solution verifying the outgoing Sommerfeld radiation condition.

Meromorphic extension of $R(z)$

The cut-off resolvent $\chi R(z)\chi$ can be meromorphically extended from $\mathbb{C}_+$ into the region

$$\mathcal{O} = \mathbb{C} \setminus \{z;\ \operatorname{Re} z \geq C|\operatorname{Im} z|^\tau,\ \operatorname{Im} z \leq -\delta \operatorname{Re} z\}, \quad \tau > 0.$$

Quantum resonances located in $]0, +\infty[$ are outgoing positive resonances of H and those in $\mathbb{C}_+ \cup] - \infty, 0[$ are eigenvalues of H. Incoming positive resonances are invisibles in the meromorphic extension of $\chi R(z)\chi$ from $\mathbb{C}_+$.

Threshold eigenvalue in non-selfadjoint case

Assume that zero is an eigenvalue of H. Then -1 is an eigenvalue of compact operator $R_0(0)W$. Let Π_{-1} denote the Riesz projection of eigenvalue -1 of $R_0(0)W$. Set

$$\omega(z) = \det(\Pi_{-1}(1 + R_0(z)W)\Pi_{-1}), z \notin \sigma(H_0).$$

$\omega(z)$ is a Gevrey function up to $z = 0$. Assume that there exist $k \in \mathbb{N}^*$ and $\omega_k \neq 0$ such that

$$\omega(z) = \omega_k z^k + O(|z|^{k+1}), \tag{8}$$

for z near 0 and $\text{Re } z < 0$.

If H is selfadjoint, (8) is always satisfied with k the multiplicity of eigenvalue zero of H.

Sub-exponential time-decay estimates

Theorem 5

(a). The set of quantum resonances of H in $\mathcal{O}$ is at most finite. In particular, the numbers of complex eigenvalues of H in $\mathcal{O}$ and of outgoing resonances $r_+(H)$ of H are at most finite.

(b). If zero is an eigenvalue of H, suppose in addition Condition (8) is satisfied. Then there exists $c > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$,

$$\|\chi(e^{-itH} - \sum_{\lambda \in \sigma_d(H) \cap \overline{\mathbb{C}}_+} e^{-itH}\Pi_\lambda - \Pi_0(t) - \sum_{\nu \in r_+(H)} e^{-it\nu} P_\nu(t))\chi\| \leq C_\chi e^{-c\, t^\beta},$$

for $t > 0$. Here $\beta = \frac{1-\mu}{1+\mu}$ and $c > 0$ is independent of χ.

Sub-exponential time-decay estimates

In the above Theorem, Π_λ is the Riesz projection associated with the discrete eigenvalue λ of H, $P_0(t)$ is contribution from zero eigenvalue

$$\Pi_0(t) = \sum_{j=0}^{k-1} t^j \Pi_{0,j},$$

with Rank $\Pi_{0,j} \leq m$, m being the algebraic multiplicity of -1 as eigenvalue of $R_0(0)W$. $P_\nu(t)$ is polynomial in t with coefficients of rank $\leq m_+(\nu)$.

Non-trivial question. How to check the condition (8) and to calculate more explicitly contributions from threshold eigenvalue and positive resonances ?

Sub-exponential time-decay estimates

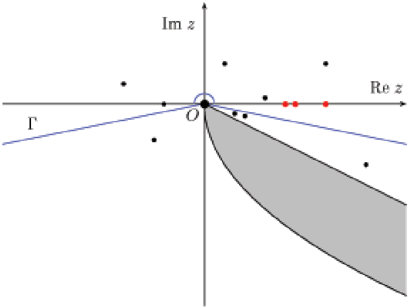

Figure: Meromorphic extension of $\chi R(z)\chi$ and the contour Γ used for e^{-itH}.

Sub-exponential time-decay estimates

The main step in the proof of Theorem 5 is to prove the resolvent $R(z) = (H - z)^{-1}$ admits an expansion of the form

$$R(z) = \sum_{j=1}^{k} z^{-j} R_{-j} + r_0(z)$$

with $\chi r_0(z) \chi \in \mathcal{G}^{(1+\gamma)}$ in $\{|z| < \delta, -\delta < \arg z < \pi + \delta\}$.

Comments on Condition (8)

Assume that zero eigenvalue of H is geometrically simple. Then

- If there exists an associated eigenfunction φ_0 of H such that

$$\int_{\mathbb{R}^n} (\varphi_0(x))^2 dx = 1, \tag{9}$$

then Condition (8) is satisfied with $k = 1$ and one has

$$\Pi_0(t) = \Pi_{0,0} = \langle \cdot, J\varphi_0 \rangle \varphi_0.$$

Here J is the complex conjugation $J : f(x) \to \overline{f(x)}$.

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  Here J is the complex conjugation $J : f(x) \to \overline{f(x)}$.
- If (9) is not satisfied and if Condition (8) is satisfied for some k, then $k \geq 2$ and $\Pi_{0,k-1}$ is of rank one and is given by

$$\Pi_{0,k-1} = \langle \cdot, J\psi_0 \rangle \psi_0$$

  for some eigenfunction ψ_0 associated with zero eigenvalue of H.

Threshold spectral analysis

Eigenvalue -1 of $R_0(0)W$ is always *semisimple* if H is selfadjoint. It is no longer the case if H is non-selfadjoint.

Let p be the algebraic multiplicity of eigenvalue -1 of $R_0(0)W$. The point of calculation is to show that there exists a nice basis b for the range of π_{-1} s.t.

$$\mathcal{M}_b(1 + R_0(z)W) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} + z \begin{pmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1p} \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{p1} & \cdots & \cdots & \cdots & a_{pp} \end{pmatrix} + O(z^2)$$

where a_{p1} is related to (9). Assumption (8) is satisfied with $k = 1$ iff $a_{p1} \neq 0$.

Comments on Condition (8)

A recent result of Maha Aafarani (PhD student, Nantes) implies that if zero eigenvalue of H is of multiplicity m and if there exists a basis $\{\varphi_1, \cdots, \varphi_m\}$ of eigenfunctions such that

$$\det(\langle \varphi_j, J\varphi_k \rangle)_{1 \le j,k \le m} \neq 0, \tag{10}$$

where J is complex conjugaison, then Condition (8) is verified with $k = m$ and one has $\Pi_{0,j} = 0$, $j = 1, \cdots, m-1$ and $\Pi_{0,0}$ is given by

$$\Pi_{0,0} = \sum_{j=1}^{m} \langle \cdot, J\psi_j \rangle \psi_j$$

where $\{\psi_j; j = 1, \cdots, m\}$ is a basis of the eigenspace of H associated with eigenvalue zero verifying

$$(\psi_i, J\psi_j) = \delta_{ij}.$$

A three-body operator

Consider now N-body problems. To simplify notation, consider only atomic-type three-body Schrödinger operators

$$H = -\Delta_{x_1} - \Delta_{x_2} + V_1(x_1) + V_2(x_2) + V_3(x_1 - x_2), \quad x_j \in \mathbb{R}^n$$

where $V_j(y)$ is real, relatively compact with respect to $-\Delta_y$ in $L^2(\mathbb{R}^n)$ and there exists some constant $\mu > 0$ such that

$$|\partial_y^\alpha V_j(y)| \leq C_\alpha |y|^{-2\mu - |\alpha|} \tag{11}$$

for y outside some compact of $\mathbb{R}^n$ and for $|\alpha| \leq 2$. This is the Hamiltonian for a quantum system consisting of three particles numbered by $\{0, 1, 2\}$, where particle 0 is of infinite mass.

A three-body operator

An example is the Coulomb Hamiltonian

$$H = -\Delta_{x_1} - \Delta_{x_2} + \frac{q_1}{|x_1|} + \frac{q_2}{|x_2|} + \frac{q_3}{|x_1 - x_2|}, \quad x_j \in \mathbb{R}^3.$$

If $q_j > 0$ for $j = 1, 2, 3$, then H verifies the weighted coercive condition and Theorem 2 holds. For an atomic-type three-body operator, it is natural to assume that $q_j < 0$ for $j = 1, 2$ and $q_3 > 0$. We want to show that the method used before can be modified to establish Gevrey estimate of the resolvent near the lowest threshold under the condition that

$$q_3 + \max\{q_1, q_2\} > 0.$$

A three-body operator

Let H be a general atomic type three-body operator with $V_j(y)$ verifying (11). Let H_j, $j = 1, 2, 3$, be subhamiltonians of H corresponding to two-cluster decomposition:

$$a_1 = \{(01)(2)\}, a_2 = \{(02)(1)\}, a_3 = \{(12)(0)\}.$$

In Jacobi coordinates,

$$H_j = -\Delta_{x_j} + V_j(x_j), \quad j = 1, 2, \quad H_3 = -2\Delta_y + V_3(y),$$

where $y = x_1 - x_2$.

A three-body operator

One has

$$e_0 = \inf \sigma_{ess}(H) = \min_{j=1,2,3} \inf \sigma(H_j).$$

Assume that e_0 is a *unique two-cluster* threshold of H, say

$$e_0 = \inf \sigma(H_1) < 0, \quad \inf \sigma(H_j) > e_0, \ j = 2, 3. \tag{12}$$

Let $\varphi_0(x_1)$ be a normalized eigenfunction of $H_1 = -\Delta_{x_1} + V_1(x_1)$ associated with e_0 and

$$I(x_2) = \langle I_1 \varphi_0, \varphi_0 \rangle_{L^2(\mathbb{R}^n_{x_1})}$$

where $I_1(x_1, x_2) = V_2(x_2) + V_3(x_1 - x_2)$.

Results

Theorem 6

Denote $x = x_2 \in \mathbb{R}^n$. *Assume that* e_0 *is a unique two-cluster threshold and there exist some constants* $c > 0$ *and* $\mu \in]0,1[$

$$I(x) \geq \frac{c}{|x|^{2\mu}} \tag{13}$$

for $|x| >> 1$. *Then H has at most a finite number of eigenvalues in* $]-\infty, e_0[$ *and*

$$R(z) = -\frac{\Pi_{e_0}}{z - e_0} + R_1(z) \tag{14}$$

where $e^{-a\langle x\rangle^{1-\mu}} R_1(z) \in \mathcal{G}^{(\frac{1+\mu}{1-\mu})}(\Omega)$.

Results

Here $\Omega = \{z \in \mathbb{C}; |z - e_0| < \delta, |\arg(z - e_0)| > \delta > 0\}$. If e_0 is an eigenvalue of H, Π_{e_0} is the associated eigenprojection; otherwise $\Pi_{e_0} = 0$. In atomic-type three-body Coulomb problem, one has

$$I(x) = \frac{q_2 + q_3}{|x|} + O(|x|^{-2}), \quad |x| >> 1.$$

Therefore all conditions of Theorem 6 are satisfied with $\mu = \frac{1}{2}$ if $q_1 < q_2 < 0$ and $q_3 + q_2 > 0$.

Remark *The case* $q_1 = q_2 < 0$ *and* $q_3 + q_2 > 0$ *can also be treated by the method of E.Skibsted-W. for spectral analysis near arbitrary two-cluster thresholds. In this case,* $E_{-+}(z)$ *is a* 3×3 *matrix-valued operator.*

Results

From Theorem 6, we obtain large-time expansion for solutions to the associated heat equation.

Corollary 7

Under the conditions of Theorem 6, for any $a > 0$, there exist some constants $C, c > 0$ such that

$$\|e^{-a\langle x\rangle^{1-\mu}}(e^{-tH} - \sum_{\lambda\in\sigma_d(H)} e^{-t\lambda}\Pi_\lambda + e^{-te_0}\Pi_{e_0})u\| \le Ce^{-te_0 - ct^\beta}\|u\|$$

for any $u \in L^2(\mathbb{R}^{2n})$.

A reduction by Grushin method

The main task for the proof of Theorem 6 is to show Theorem 2 still holds when the model resolvent $(H_0 - \lambda)^{-1}$ is replaced by $(H_0 + W_\infty(\lambda) - \lambda)^{-1}$ where $W_\infty(\lambda)$ is a non-local operator arising from Grushin reduction.

Let φ_0 be a normalized eigenfunction of H_1 associated with e_0 and $F = \{\varphi_0(x_1)f(x_2); f \in L^2(\mathbb{R}^n)\}$. Denote $\Pi : L^2(\mathbb{R}^{2n}) \to F$ the orthogonal projection and $\Pi' = 1 - \Pi$. Let $H' = \Pi' H \Pi'$. Then

$$\inf \sigma_{ess}(H') > e_0,$$

(see W., 2004). There are two cases:

$$(a). \quad e_0 \notin \sigma_d(H'); \quad (b). \quad e_0 \in \sigma_d(H').$$

A reduction by Grushin method

Consider the simpler case $e_0 \notin \sigma_d(H')$. Then $R'(z) = (H' - z)^{-1}\Pi'$ is holomorphic for z near e_0. By Grushin method, one can establish a representation formula for $R(z) = (H - z)^{-1}$:

$$R(z) = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z) \tag{15}$$

where

$$\begin{aligned} E(z) &= R'(z), \\ E_+(z) &= -R'(z)I_1(\varphi_0 \otimes \cdot) + (\varphi_0 \otimes \cdot), \\ E_-(z) &= (\cdot, \varphi_0)_1 - (\varphi_0, I_1 R'(z)\cdot)_1, \\ E_{-+}(z) &= (z - e_0) - (-\triangle_{x_2} + I(x_2)) + (I_1 R'(z) I_1(\varphi_0 \otimes \cdot), \varphi_0)_1. \end{aligned}$$

where $I_1 = V_2(x_2) + V_3(x_1 - x_2)$, $I(x_2) = (I_1\varphi_0, \varphi_0)_1$ and $(\cdot, \cdot)_1$ denotes scalar product in x_1 variables.

A reduction by Grushin method

$E(z)$ and $E_\pm(z)$ are holomorphic for z near e_0. We need to study $E_{-+}(z)^{-1}$. Set $\lambda = z - e_0$ and $x = x_2$. Write $E_{-+}(z)$ as

$$E_{-+}(e_0 + \lambda) = \lambda - (-\Delta_x + V(\lambda)),$$

where $V(\lambda) = I(x) + W(\lambda)$ with

$$W(\lambda) = (\varphi_0, I_1 R'(\lambda + e_0) I_1(\varphi_0 \otimes \cdot))_1. \tag{16}$$

$W(\lambda)$ is holomorphic for λ near 0 and

$$\|\langle x\rangle^{1+2\mu} W(\lambda) \langle x\rangle^{1+2\mu}\| \le C$$

uniformly in λ near 0, because $\Pi I_1 \Pi' = O(\langle x\rangle^{-1-2\mu})$.

Gevrey estimates for $E_{-+}(\lambda + e_0)^{-1}$

To simplify presentation, assume that e_0 is not an eigenvalue of H. Then 0 is not an eigenvalue of $E_{-+}(e_0)$. We want to prove Gevrey estimates for $E_{-+}(e_0 + \lambda)^{-1}$. If the term $W(\lambda)$ were absent, under the condition (13) we can split $-\Delta + I(x)$ as

$$-\Delta + I(x) = H_0 + U(x)$$

where H_0 verifies condition of Theorem 2 and $U(x)$ is of compact support. Gevrey estimates for $(-\Delta + I - \lambda)^{-1}$ follows from Theorem 2 and the equation

$$(-\Delta + I - \lambda)^{-1} = R_0(\lambda)(1 + UR_0(\lambda))^{-1},$$

if zero is not eigenvalue of $-\Delta + I$. This argument can not be applied to $E_{-+}(\lambda + e_0)^{-1}$, because $W(\lambda)$ is non-local and has no sufficient decay.

Gevrey estimates for $E_{-+}(\lambda + e_0)^{-1}$

To treat the term $W(\lambda)$, we modify the proof of Theorem 2 from the very beginning and exploit the analyticity of $W(\lambda)$ in λ. Let φ_s be the weight function used in Lemma 1: $\varphi_s(x) = (1 + \frac{|x|^2}{R_s^2})^{s/2}$ where $R_s = M\langle s\rangle^{\frac{1}{1-\mu}}$.

Lemma 8

For $r, r' \in \mathbb{R}$ with $r + r' \leq 2 + 4\mu$, one has

$$\|\langle x\rangle^r \varphi_s W(\lambda)\varphi_{-s}\langle x\rangle^{r'}\| \leq C$$

uniformly in $s \in \mathbb{R}$ and $|\lambda| < \delta$.

Gevrey estimates for $E_{-+}(\lambda + e_0)^{-1}$

Proof. Lemma 8 follows from

$$\varphi_s H' \varphi_{-s} = H' + O(\frac{s}{M\langle s\rangle^{\frac{1}{1-\mu}}}) = H' + O(\frac{1}{M})$$

uniformly in s and from the fact e_0 belongs to the resolvent set of H'.
□

Let $\chi_R(x) = \chi_1(\frac{x}{R})$, $R \geq 1$, where $\chi_1 \in C^\infty(\mathbb{R}^n)$ such that $\chi_1(x) = 0$ if $|x| \leq 1$ and $\chi_1(x) = 1$ if $|x| \geq 2$. Set

$$F(\lambda) = -\Delta_x + 1 - \chi_R(x) + \chi_R V(\lambda). \qquad (17)$$

$F(\lambda) = H_0 + \chi_R W(\lambda)$ with $H_0 = -\Delta + 1 - \chi_R(x) + \chi_R I(x)$ verifying conditions of Lemma 1.

Gevrey estimates for $E_{-+}(\lambda + e_0)^{-1}$

Lemma 9

Assume (13) for some $\mu \in]0,1[$. There exist some constants $M, R > 0$ such that

$$\|\langle x\rangle^{-\mu}\varphi_s u\| + \|\nabla(\varphi_s u)\| \leq C\|\langle x\rangle^{\mu}\varphi_s(F(\lambda) - \lambda)u\|$$

uniformly in $s \in \mathbb{R}$, $\lambda \in]-\delta, 0]$ and $u \in \mathcal{S}$.

Lemma 9 follows from Lemma 1 applied to H_0 and Lemma 8. In the following, $R > 1$ is fixed such that Lemma 9 holds.

Gevrey estimates for $E_{-+}(\lambda + e_0)^{-1}$

Lemma 9 shows that $G(\lambda) = (F(\lambda) - \lambda)^{-1}$ verifies

$$\|\langle x\rangle^{-\mu}\varphi_s G(\lambda)\varphi_{-s}\langle x\rangle^{-\mu}\| + \|\nabla\varphi_s G(\lambda)\varphi_{-s}\langle x\rangle^{-\mu}\| \le C$$

uniformly in $\lambda \in]-\delta, 0]$ and $s \in \mathbb{R}$. By method of commutator, we obtain

$$\|\langle x\rangle^{-2\mu}\varphi_s G(\lambda)\varphi_{-s}\| \le C \tag{18}$$

uniformly in s and λ.

Gevrey estimates for $E_{-+}(\lambda + e_0)^{-1}$

A heavy task in the proof of Theorem 6 is to show the following

Theorem 10

There exist $C, \delta > 0$ *such that for any* $r \geq 0$, $N \in \mathbb{N}$ *and* $\lambda \in]-\delta, 0]$, *one has*

$$\begin{aligned} &\| \langle x \rangle^{-2\mu} \langle x_{N,r} \rangle^{-(2N+r)\mu} G^{(N)}(\lambda) \langle x_{N,r} \rangle^{r\mu} \| \\ &\leq C^{N+1} N! \langle (2N+r)\mu \rangle^{\gamma N} \end{aligned} \tag{19}$$

Here

$$x_{N,r} = \frac{x}{R_{N,r}} \text{ with } R_{N,r} = M \langle (2N+r)\mu \rangle^{\frac{1}{1-\mu}}$$

and $\langle x_{N,r} \rangle = (1 + |x_{N,r}|^2)^{\frac{1}{2}}$.

Proof of Theorem 10

Proof. The case $N = 0$ and $r \geq 0$ of Theorem 10 follows from (18) with $s = r\mu$. In the general case $N \geq 0$ and $r \geq 0$, using

$$G^{(N+1)}(\lambda) = (G(\lambda)^2 - G(\lambda)\chi_R W'(\lambda)G(\lambda))^{(N)},$$

we prove by induction

$$\|\langle x\rangle^{-2\mu}\langle x_{N,r}\rangle^{-(2N+r)\mu} G^{(N)}(\lambda)\langle x_{N,r}\rangle^{r\mu}\|$$

$$\leq C_N^{N+1} N!\langle(2N+r)\mu\rangle^{\gamma N}$$

with $C_N \leq C_{N+1} \leq C_N(1 + \frac{c}{N^{1+\gamma}})$. □

Proof of Theorem 6

Note

$$\|\langle x_{N,r}\rangle^{(2N+r)\mu} e^{-a\langle x\rangle^{1-\mu}}\|_{L^\infty} \le A_a^{N+r}.$$

Theorem 10 gives

$$\|\langle x\rangle^{-\tau} e^{-a\langle x\rangle^{1-\mu}} G^{(N)}(\lambda)\langle x\rangle^{\tau}\| \le C_a^{N+1+\tau} N!\langle N+\tau\rangle^{\gamma N+\frac{\tau}{1-\mu}}$$

for $\tau \ge 0$ and $N \in \mathbb{N}$. Theorem 6 in the case $e_0 \notin \sigma_p(H) \cup \sigma_d(H')$ is deduced from the above estimate and

$$\begin{aligned} R(z) &= E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z) \\ E_{-+}(\lambda+e_0)^{-1} &= -G(\lambda)(1+U(\lambda)G(\lambda))^{-1}, \end{aligned}$$

where $U(\lambda) = (1-\chi_R)(V(\lambda)-1)$. Note that $U(\lambda)e^{a\langle x\rangle^{1-\mu}}$ is bounded if $a > 0$ is small. □

Happy birthday André !